Painlevé expressions for LOE, LSE and interpolating ensembles

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Abstract

We consider an ensemble which interpolates the Laguerre orthogonal ensemble and the Laguerre symplectic ensemble. This interpolating ensemble was introduced earlier by the author and Rains in connection with a last passage percolation model with a symmetry condition. In this paper, we obtain a Painelevé V expression for the distribution of the rightmost particle of the interpolating ensemble. Special cases of this result yield the Painlevé V expressions for the largest eigenvalues of Laguerre orthogonal ensemble and Laguerre symplectic ensemble of finite size.

1 Introduction

Given a sequence $\xi = (\xi_1, \cdots, \xi_N)$, the Vandermonde product of ξ is denoted by

$$\Delta_N(\xi) = \prod_{1\le i<j\le N} (\xi_i - \xi_j). \tag{1.1}$$

For any real constant $A > -1/2$, we consider the probability density function defined by

$$p(\xi_1, \cdots, \xi_N; A) = Z_{N,A}^{-1} \Delta_N(\xi) \prod_{j=1}^{N} e^{-\frac{1}{2}\xi_j} e^{A(-1)^j \xi_j} \tag{1.2}$$

on the ordered set $\{0 \le \xi_N \le \cdots \le \xi_1\}$, where

$$Z_{N,A} = (A + \frac{1}{2})^{-N} \prod_{j=0}^{N-1} j! \tag{1.3}$$

is the normalization constant. The main purpose of this paper is to express the distribution of the rightmost 'particle' ξ_1 in terms of a solution of the Painlevé V equation (see Theorem 1.3 below).

There are two reasons that we are interested in the above density. The first is that (1.2) interpolates the Laguerre orthogonal ensemble and the Laguerre symplectic ensemble in the random matrix theory,

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as A varies from 0 to $+\infty$. Letting $A \to 0$ or $A \to \infty$ in the Painlevé expression, we verify in particular that the distribution function for the rightmost particle ξ_1 satisfies the Painlevé V equation for the Laguerre orthogonal and symplectic ensembles, respectively. These are new results in the literature. The second is that the above density function represents the probability distribution of a certain last passage percolation model with a symmetry condition. Indeed, the density (1.2) was introduced in [5] (see Remark 7.6.1) as a formula for the distribution of the last passage time in this percolation model. We now discuss these two aspects of the above density function.

Interpolating ensemble

We first discuss the connection to the Laguerre ensembles. Let $w(x)$ be a weight function on $\mathbb{R}$ or on a subset of $\mathbb{R}$ which decays sufficiently fast as $x \to \pm\infty$. Consider the density function

$$\frac{1}{Z}\Delta_N(\xi)^\beta \prod_{j=1}^{N} w(\xi_j) \tag{1.4}$$

on the set $\xi_N \le \cdots \le \xi_1$, where $\beta > 0$ is fixed, and Z is the normalization constant. The ensemble with the special choice of weight function

$$w(x) = x^\alpha e^{-\frac{1}{2}x} 1_{x \ge 0} \tag{1.5}$$

is called the Laguerre orthogonal ensemble (LOE), Laguerre unitary ensemble (LUE), and Laguerre symplectic ensemble (LSE) for $\beta = 1, 2$ and 4, respectively (see e.g. [22]). The Laguerre ensembles are of basic interest in the multivariate analysis of statistics (see, e.g. [20]). Especially, the LOE with $\alpha = M - 1 - N$ represents the probability of the principal components (i.e. the singular values) of an $M \times N$ random matrix X whose entries are independent (real) Gaussian random variables of mean 0, variance 1.

Introduce the density function, for real A, given by

$$p(\xi_1, \cdots, \xi_N; A; w; \beta) := Z^{-1}_{N,A;w;\beta}\Delta_N(\xi)^\beta \prod_{j=1}^{N} w(\xi_j) e^{A(-1)^j \xi_j} \tag{1.6}$$

on the set $\mathbb{R}^N_{ord} := \{\xi_N \le \xi_{N-1} \le \cdots \le \xi_1\}$. This density function generalizes and also interpolates the ensembles (1.4) for different β's in the following sense.

Proposition 1.1. *When $A = 0$,*

$$p(\xi_1, \cdots, \xi_N; 0; w; \beta) = Z^{-1}_{N,0;w}\Delta_N(\xi)^\beta \prod_{j=1}^{N} w(\xi_j) \tag{1.7}$$

with some constant $Z(N, 0; w)$. Let $w(x) = e^{-V(x)}$ be a positive, C^1-function supported on a subset of $\mathbb{R}$ such that $V(x) \ge c_0|x|$ for some $c_0 > 0$ as $x \to \pm\infty$, and $\|V'\|_{L^\infty} \le C_0$. Assuming that N is even,

for a bounded uniformly continuous function $f(\xi_1, \cdots, \xi_N)$*, we have*

$$\lim_{A\to+\infty}\int_{\mathbb{R}^N_{ord}} f(\xi_1,\cdots,\xi_N)p(\xi_1,\cdots,\xi_N;A;w;\beta)d\xi_1\cdots d\xi_N$$
$$= Z^{-1}_{N,\infty;w;\beta}\int_{\mathbb{R}^{N/2}_{ord}} g(\zeta_1,\cdots,\zeta_{N/2})\Delta_{N/2}(\zeta)^{4\beta}\prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j, \tag{1.8}$$

where $g(\zeta_1,\zeta_2,\cdots,\zeta_{N/2}) := f(\zeta_1,\zeta_1,\zeta_2,\zeta_2,\cdots,\zeta_{N/2},\zeta_{N/2})$ *and*

$$Z_{N,\infty;w;\beta} := \int_{\mathbb{R}^{N/2}_{ord}} \Delta_{N/2}(\zeta)^{4\beta}\prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j. \tag{1.9}$$

We also have for any $t\in\mathbb{R}$*,*

$$\lim_{A\to+\infty}\int_{\mathbb{R}^N_{ord}\cap\{\xi_1\le t\}} p(\xi_1,\cdots,\xi_N;A;w;\beta)d\xi_1\cdots d\xi_N$$
$$= Z^{-1}_{N,\infty;w;\beta}\int_{\mathbb{R}^{N/2}_{ord}\cap\{\zeta_1\le t\}} \Delta(\zeta)^{4\beta}\prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j. \tag{1.10}$$

The case $A = 0$ is trivial from the expression (1.6). An explanation for the change $\beta \to 4\beta$ in the case when $A \to +\infty$ is the following. We first note that the term involving A in (1.6) is

$$e^{-A(\xi_1-\xi_2+\xi_3-\xi_4+\cdots)}. \tag{1.11}$$

Since ξ_j's are ordered, $\xi_1 \ge \xi_2 \ge \cdots \ge \xi_N$ (N even), the term $\xi_1 - \xi_2 + \xi_3 - \xi_4 + \cdots$ is always non-negative. Thus as $A \to +\infty$, we have a non-trivial limit for (1.6) only when $\xi_1 = \xi_2$, $\xi_3 = \xi_4$, $\cdots$ (the term $Z^{-1}_{N,A;w;\beta}$ grows polynomially in A ; see Lemma 2.1 below). If we set $\zeta_1 := \xi_1 = \xi_2$, $\zeta_2 := \xi_3 = \xi_4$, $\cdots$, simple algebra shows that the Vandermonde term $\Delta_N(\xi)$ in (1.6) becomes $\Delta_{N/2}(\zeta)^4$ if one drops the terms $(\xi_{2k-1} - \xi_{2k})$, $1 \le k \le N/2$, which vanish. (See also the Remark in Section 2 below for changes $\beta \to k^2\beta$, $k \in \mathbb{N}$). The full proof of this Proposition is given in Section 2.

In view of the Coulomb gas interpretation of random matrix theory (see e.g. [22]), the term involving A in the density (1.6) represents the pairwise-attraction of particles, in addition to the log repulsion given by $\Delta_N(\xi)$ and the "external field" w. As A becomes large, the pairwise-attraction becomes stronger, and for each $j = 1,\cdots,N/2$, the pair of particles ξ_{2j-1}, ξ_{2j} gets closer, and eventually sticks together to form one particle ζ_j.

If we take $w(x) = e^{-\frac{1}{2}x}$ on $x \ge 0$ and $\beta = 1$ in the above Proposition, (1.2) becomes the density for LOE (1.5) with $\alpha = 0$ when $A = 0$, while the limit of (1.2) as $A \to +\infty$ is, with the modification $e^{-\frac{1}{2}\xi_j}$ by $e^{-\xi_j}$, the density for LSE (1.5) with $\alpha = 0$. Thus by taking $A = 0$ and $A = \infty$, the Painlevé V expression (Theorem 1.3 below) for the largest particle of the interpolating ensemble (1.2) yields the Painlevé V expressions for the largest eigenvalue of LOE and LSE (see Corollary 1.4 below) with $\alpha = 0$ for any *finite* $N \in \mathbb{N}$.

In the random matrix context, there have been many works that express the various distributions of matrix ensembles in terms of differential equations (see e.g. [30] and the references therein). Most

of results are for the unitary ensemble ($\beta = 2$), and there are relatively few results for $\beta = 1$ and $\beta = 4$ ([28, 14, 13, 17, 20, 16]). For the Laguerre ensembles, the probability distributions for $\beta = 2$ are found to be expressible in terms of Painlevé V equation ([26]) for any finite N, and in terms of the Painlevé II equation ([27]) in the limiting case $N \to \infty$. On the other hand, for $\beta = 1$, Johnstone ([20]) recently analyzed the limiting case $N \to \infty$, and obtained a Painlevé II expression. The Painlevé V expressions for LOE/LSE for *finite* N in this paper are new. While this paper was being written, Forrester and Witte obtained different formulas for the largest eigenvalues of LOE and LSE ([16]). Their formulas involve Painlevé III' systems instead of Painlevé V. The relationship between these two formulas is an intriguing question and remains unclear.

Last passage percolation with a symmetry condition and the totally asymmetric exclusion process with symmetry condition

As mentioned earlier, the density function (1.2) also arises in connection with a last passage percolation model. For $r > 0$, let $e(r)$ denote the exponentially distributed random variable with mean r : the density function of $e(r)$ is $r^{-1}e^{-x/r}$ for $x > 0$ and 0 for $x \le 0$. By $e(0)$ we understand the random variable identically equal to 0. Fix $\rho \ge 0$. To each site $(i, j) \in \mathbb{Z}_+^2$, we attach a random variable $u(i, j)$ taken as follows :

$$u(i,j) \sim e(1), \qquad i < j, \tag{1.12}$$

$$u(i,i) \sim e(\rho), \tag{1.13}$$

$$u(j,i) = u(j,i), \qquad i < j. \tag{1.14}$$

Except for the symmetry condition $u(i, j) = u(j, i)$, the random variables are independent. Note that the condition (1.14) implies the symmetry of the configuration of random variables in $\mathbb{Z}_+^2$ about the line $y = x$. An up/right path π is a collection of sites $\{(i_k, j_k)\}_{k=1}^r$ satisfying $(i_{k+1}, j_{k+1}) - (i_k, j_k) = (1, 0)$ or $(0, 1)$. Let $\Pi(N)$ be the set of up/right path π from $(1, 1)$ to (N, N). Define the random variable

$$H^{\boxslash}(N;\rho) = \max\Big\{ \sum_{(i,j)\in\pi} u(i,j) : \pi \in \Pi(N) \Big\}. \tag{1.15}$$

If one interprets $u(i, j)$ as the (random) passage time to pass through the site (i, j), $H^{\boxslash}(N; \rho)$ is the last passage time to go from $(1, 1)$ to (N, N) along a directed (up/right) path. The relation of $H^{\boxslash}(N; \rho)$ to the above interpolating density function is the following.

Proposition 1.2. *(Remark 7.6.1 [5]) We have*

$$\mathbb{P}(H^{\boxslash}(N;\rho) \le x) = \int_{0\le\xi_N\le\cdots\le\xi_1\le x} p(\xi_1, \cdots, \xi_N; \frac{1}{\rho} - \frac{1}{2}) \prod_{j=1}^{N} d\xi_j. \tag{1.16}$$

This Proposition is stated without full proof in Remark 7.6.1 [5]: a full proof if given in Section 3 below.

Closely related is a one-dimensional interacting particle system, the totally asymmetric simple exclusion process (TASEP) ([21, 19]). TASEP is a continuous-time stochastic process on the integer lattice $\mathbb{Z}$. At any time, each site is either occupied by a particle or empty. If a particle is at a site j and its right-hand-site $j+1$ is empty, the particle jumps to its right-hand-site after a random waiting time given by an exponential random variable of mean 1. Thus the particles move only to the right. The waiting time for the jumps is independent and identically distributed at each site and each (continuous) time. These rules describe the usual totally asymmetric simple exclusion process (see e.g. [19], [24], [21]). In [19], Johansson showed that the TASEP with special initial data (all negative sites are occupied and all non-negative sites are empty) can be mapped to the last passage percolation above *without* the symmetry condition (1.14). This mapping was further generalized in [24] for TASEP with random Bernoulli initial data.

The above last passage percolation model with the symmetry condition (1.14) is also related to a TASEP, but now the process is defined only on the non-negative integer lattice, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For the sites $j > 0$, the jump rules remain the same as before, but we assume that there is a creation process at the origin $j = 0:$ when the site $j = 0$ is empty, a particle is created after a random exponential waiting time of mean ρ. For initial data, we assume that all the sites are empty. Then one can show that ([24]) the number of particles $N(t)$ that have been created at the origin up to time t satisfies

$$\mathbb{P}(N(t) \leq N) = \mathbb{P}(H^{\boxslash}(N;\rho) > t). \tag{1.17}$$

In the last passage percolation model above, one might also be interested in the last passage time from $(1,1)$ to (M,N) for general $M \neq N$. In terms of TASEP, this is equivalent to the number of particles that have jumped across the site $M-N$. But there is yet no formula like Proposition 1.2 for general $M \neq N$.

Results

Now we state the main results. From Proposition 1.2, the following results imply the Painlevé V expressions for the distribution of the rightmost 'particle' from the interpolating ensemble (1.2).

Theorem 1.3. *With the notation $A = \frac{1}{\rho} - \frac{1}{2}$, and $w = \frac{2}{\rho} - 1 = 2A$, we have for $x > 0, \rho > 0, N \in \mathbb{N}$,*

$$\mathbb{P}(H^{\boxslash}(N;\rho) \leq x) = \frac{1}{2}\Big\{ \big[\mathrm{a_N}(x,\rho) - \mathrm{b_N}(x,\rho)\big] \big(\mathrm{E_N}(x)\big)^{-1} + \big[\mathrm{a_N}(x,\rho) + \mathrm{b_N}(x,\rho)\big]\,\mathrm{E_N}(x) \Big\}\,\mathrm{F_N}(x) \tag{1.18}$$

for some functions

$$\mathrm{F_N}(x) := \exp\Big\{ \int_x^\infty \frac{1}{4}\alpha(y;N)dy \Big\}, \tag{1.19}$$

$$\mathrm{E_N}(x) := \exp\Big\{ \int_x^\infty \frac{1}{4}\beta(y;N)dy \Big\} \tag{1.20}$$

and $\mathrm{a_N}(x,\rho), \mathrm{b_N}(x,\rho)$. The functions $\alpha(x), \beta(x), \mathrm{a_N}(x,\rho), \mathrm{b_N}(x,\rho)$ are analytic in $x > 0, \rho > 0$, and satisfy the following properties :

(i) (Painlevé V) The functions $\alpha(x;N), \beta(x;N)$ *as functions in* x *satisfy*

$$\alpha'(x;N) = \frac{1}{2}(\beta(x;N))^2. \tag{1.21}$$

The function $\theta(x) = \theta(x;N) := -\frac{1}{2}x\alpha(x)$ *solves the Painlevé V equation*

$$(x\theta'')^2 = (\theta - x\theta')(\theta - x\theta' + 4(\theta')^2 + 4N\theta'). \tag{1.22}$$

(ii) (Asymptotics of α, β *as* $x \to \infty$*) Fix* $0 < \epsilon < 1/4$*. For each fixed* $N \in \mathbb{N}$*, as* $x \to +\infty$*,*

$$\beta(x) = 2(-1)^N L_{N-1}^{(1)}(x)e^{-\frac{1}{2}x} + O(e^{-(1-\epsilon)x}) = \frac{-2x^{N-1}}{(N-1)!}e^{-\frac{1}{2}x}(1 + O(x^{-1})) \tag{1.23}$$

and

$$\alpha(x) = \int_{\infty}^{x} 2(L_{N-1}^{(1)}(y))^2 e^{-y}dy + O(e^{-\frac{3}{2}(1-\epsilon)x}) = \frac{-2x^{2N-2}}{((N-1)!)^2}e^{-x}(1 + O(x^{-1})), \tag{1.24}$$

where $L_{N-1}^{(1)}(x)$ *is the Laguerre polynomial of degree* $N-1$ *with parameter* 1 *; (see, e.g. [1])*

$$L_{N-1}^{(1)}(x) = \sum_{j=0}^{N-1} \binom{N}{j+1} \frac{(-x)^j}{j!}. \tag{1.25}$$

(iii) (Lax pair for Painlevé V) The functions $\mathrm{a_N}(x,\rho)$ *and* $\mathrm{b_N}(x,\rho)$ *are real and smooth in* $x > 0$ *and* $\rho > 0$*, and they satisfy the differential equations (note* $w = \frac{2}{\rho} - 1$*)*

$$\frac{\partial}{\partial x}\begin{pmatrix} \mathrm{b_N} \\ \mathrm{a_N} \end{pmatrix} = -\frac{1}{2}w\begin{pmatrix} \mathrm{b_N} \\ 0 \end{pmatrix} + \frac{1}{2}\beta\begin{pmatrix} \mathrm{a_N} \\ \mathrm{b_N} \end{pmatrix}, \tag{1.26}$$

and

$$\frac{\partial}{\partial w}\begin{pmatrix} \mathrm{b_N} \\ \mathrm{a_N} \end{pmatrix} = -\frac{1}{2}x\begin{pmatrix} \mathrm{b_N} \\ 0 \end{pmatrix} + \frac{1}{1-w^2}\begin{pmatrix} -(x\alpha)' & (x\beta)' \\ -(x\beta)' & (x\alpha)' \end{pmatrix}\begin{pmatrix} \mathrm{b_N} \\ \mathrm{a_N} \end{pmatrix} - \frac{1}{2}\frac{w}{1-w^2}x\beta\begin{pmatrix} \mathrm{a_N} \\ \mathrm{b_N} \end{pmatrix}. \tag{1.27}$$

(iv) (Asymptotics of $\mathrm{a_N}, \mathrm{b_N}$ *as* $x \to \infty$*) For any* $0 < \epsilon < 1/4$*, as* $x \to \infty$*, we have*

$$\mathrm{a_N}(x,\rho) = \begin{cases} 1 + O(e^{-(1-2\epsilon)x}), & 0 < \rho \le \frac{2}{2-\epsilon}, \\ \Phi(w;x)(\Lambda_1(-w,x) + O(e^{-(1-2\epsilon)x})), & \rho > \frac{2}{2-\epsilon}, \end{cases} \tag{1.28}$$

$$\mathrm{b_N}(x,\rho) = \begin{cases} -\Lambda_1(w,x) + O(e^{-(1-2\epsilon)x}\Phi(w;x)), & 0 < \rho \le \frac{2}{\epsilon}, \\ -\Phi(w;x)(1 + O(e^{-(1-2\epsilon)x}), & \rho > \frac{2}{\epsilon}, \end{cases} \tag{1.29}$$

where

$$\Lambda_1(w,x,N) := \frac{1}{2\pi i}\int_{|s-1|=\frac{1}{2}|1-w|} \Phi(s;x)\frac{ds}{s-w}, \qquad \Phi(w;x) := e^{-\frac{1}{2}xw}\left(\frac{1+w}{1-w}\right)^N. \tag{1.30}$$

(v) (Asymptotics of $\mathrm{a_N}, \mathrm{b_N}$ *as* $\rho \to 0^+, 2, \infty$*) We have*

$$\lim_{\rho\to 0^+} \mathrm{a_N}(x,\rho) = 1, \qquad \lim_{\rho\to 0^+} \mathrm{b_N}(x,\rho) = 0 \tag{1.31}$$

and

$$\mathrm{a_N}(x,2) = \mathrm{E_N}(x)^2, \qquad \mathrm{b_N}(x,2) = -\,\mathrm{E_N}(x)^2. \tag{1.32}$$

Also for fixed $y > 0$*,*

$$\lim_{\rho\to\infty} \mathrm{a_N}(y\rho,\rho) = P(N,y), \qquad \lim_{\rho\to\infty} \mathrm{b_N}(y\rho,\rho) = 0, \tag{1.33}$$

where $P(N,y)$ *is the incomplete Gamma function (see, e.g. [1])*

$$P(N,y) = \frac{1}{(N-1)!}\int_0^y e^{-t}t^{N-1}dt. \tag{1.34}$$

Remark. The existence of the solution θ (hence α) to the Painlevé V equation (1.22) with the asymptotic condition (1.24) is a part of the Theorem. But as yet we do not have uniqueness for θ (or α) as a solution of (1.22) with asymptotic condition (1.24). Another missing piece of information is the asymptotics of α and β as $x \to 0^+$. These issues will be studied in a later publication.

From Proposition 1.1, by using (1.32) and (1.31), Theorem 1.3 implies the following results for LOE and LSE at the special values $\rho = 2$ $(A = 0)$ and $\rho \to 0^+$ $(A \to \infty)$.

Corollary 1.4. *We have, with the notation* $\mathbb{R}^N_{ord}(\xi) := \{\xi_N \le \xi_{N-1} \le \cdots \le \xi_1\}$*, for any* $x > 0$*,*

$$\frac{1}{2^N\prod_{j=0}^{N-1} j!}\int_{\mathbb{R}^N_{ord}(\xi)\cap\{\xi_1\le x\}} \Delta_N(\xi)\prod_{j=1}^N e^{-\frac{1}{2}\xi_j}d\xi_j = \mathbb{P}(H^{\boxslash}(N;2)\le x) = \mathrm{E_N}(x)\,\mathrm{F_N}(x), \tag{1.35}$$

and for N *even,*

$$\frac{1}{\prod_{j=0}^{N-1} j!}\int_{\mathbb{R}^{N/2}_{ord}(\zeta)\cap\{\zeta_1\le x\}} (\Delta_{N/2}(\zeta))^4\prod_{j=1}^{N/2} e^{-\zeta_j}d\zeta_j = \mathbb{P}(H^{\boxslash}(N;0)\le x) = \frac{1}{2}\Big\{\big(\mathrm{E_N}(x)\big)^{-1} + \mathrm{E_N}(x)\Big\}\,\mathrm{F_N}(x). \tag{1.36}$$

Remark. Once the uniqueness of θ (or α) is proven (see Remark above), and also the uniqueness of β is established, the above Corollary provides a tool for numerical computations of the largest eigenvalue distribution of LOE and LSE for the special case $w(x) = e^{-\frac{1}{2}x}$ (see (1.5)) for any finite N.

For LUE, the largest eigenvalue distribution was obtained by Tracy and Widom ([26]) :

$$\frac{1}{\prod_{j=0}^{N-1}(j!)^2}\int_{\mathbb{R}^N_{ord}(\xi)\cap\{\xi_1\le x\}} (\Delta_N(\xi))^2\prod_{j=1}^N e^{-\frac{1}{2}\xi_j}d\xi_j = (\mathrm{F_N}(x))^2. \tag{1.37}$$

Note the special structure of the formulas (1.35), (1.36) and (1.37), from which an interesting interrelationship of the largest eigenvalues of LOE, LUE and LSE can be derived. We refer the reader to

[14] for a full discussion on inter-relationship between orthogonal, symplectic and unitary ensembles. We also remark that Corollary 1.4 applies only for the case $\alpha = 0$ of the Laguerre weight (1.5). We do not have results for LOE and LSE of other values of α. On the other hand, LUE with different values of $\alpha \neq 0$ was analyzed in [26].

If we take the limit $\rho \to \infty$ ($A \downarrow -1/2$; note that (1.2) is defined for $A > -1/2$), we have $\mathbb{P}(H^{\boxslash}(N;\rho) \le x) \to 0$ for fixed $x > 0$. To obtain a non-trivial limit from (1.33) we set $x = y\rho$ and let $\rho \to \infty$ while $y > 0$ is fixed. Note that from (1.23) and (1.24), we have $\mathrm{F_N}(x) = 1 - \frac{1}{2((N-1)!)} x^{2N-2} e^{-x}(1 + O(x^{-1}))$ and $\mathrm{E_N}(x) = 1 - \frac{1}{(N-1)!} x^{N-1} e^{-\frac{1}{2}x}(1 + O(x^{-1}))$ as $x \to \infty$, and hence we find that $\mathrm{E_N}(x)\,\mathrm{F_N}(x) \to 1$ and $\mathrm{E_N}(x)^{-1}\,\mathrm{F_N}(x) \to 1$ as $x \to \infty$.

Corollary 1.5. *For any fixed $y > 0$,*

$$\lim_{\rho \to \infty} \mathbb{P}(H^{\boxslash}(N;\rho) \le y\rho) = P(N, y), \tag{1.38}$$

where $P(N, y)$ is the incomplete Gamma function (1.34).

This result is consistent with the intuition that when ρ is large, the longest up/right path in the percolation model is basically the diagonal line through the points (i, i), $i = 1, \cdots, N$. As $\rho \to \infty$, we expect that the random variable $H^{\boxslash}(N;\rho)$ is close to $S_N(\rho) := u(1,1) + u(2,2) + \cdots + u(N,N)$, the sum of N i.i.d. exponential random variables of mean ρ. A direct calculation shows that $\mathbb{P}(S_N(\rho) \le a) = P(N, a/\rho)$. Thus Corollary 1.5 shows that indeed $H^{\boxslash}(N;\rho) \sim S_N(\rho)$ when $\rho \to \infty$.

This paper is organized as follows. The proof of Proposition 1.1 is given in Section 2. In Section 3, we consider a different percolation model which has a geometric random variable at each site instead of an exponential random variable. Since a geometric random variable converges to an exponential random variable in an appropriate limit, the related percolation model with geometric random variables also converges to the percolation model with exponential random variables (see Lemma 3.1 below). In Section 3 and Section 4, we present two different formulas for the geometric percolation model. A multi-sum formula is given in Section 3, and in an appropriate limit, this multi-sum formula becomes the multi-integral formula in Proposition 1.2 for the exponential percolation model. In Section 4, we express the distribution for the geometric percolation model in terms of orthogonal polynomials (see Lemma 4.2 below). The appropriate limit of these orthogonal polynomials yields the Painlevé expressions, Theorem 1.3. The asymptotics of orthogonal polynomials are obtained in Section 6 by applying a steepest-descent method to the associated Riemann-Hilbert problem, and the proof of Theorem 1.3 is given at the end of Section 6. Some properties of the Riemann-Hilbert problem for the Painlevé V equation, which will be used in Section 6 are summarized in Section 5. Section 7 presents some results for the percolation model as $N \to \infty$. Finally, in Section 8 we show that the computation of Tracy and Widom ([29]) which expresses the correlation functions for orthogonal and symplectic ensembles in terms of determinants can also be applied to the density (1.6) with $\beta = 1$ which includes (1.2) as a special case.

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2 Proof of Proposition 1.1

The identity (1.7) is trivial. We prove (1.8) in this section. We basically prove that when $A \to \infty$, the function $p(\xi_1, \cdots, \xi_N; A; w)$ on $\mathbb{R}^N_{ord}$ will concentrate on the subset satisfying $\xi_{2j-1} = \xi_{2j}$ for each j. We present the proof only for the case when $\beta = 1$. But it would be clear that the proof for general $\beta > 0$ will be the same. In the below, we omit any dependence on β.

We assume that N is even and $A \geq 1$. Substitute $\xi_{2j-1} = \xi_{2j} + x_j$, $j = 1, \cdots, N/2$, and set $\zeta_j = \xi_{2j}$. We use $\zeta_1, \cdots, \zeta_{N/2}$ and $x_1, \cdots, x_{N/2}$ as new variables. Then the region of integration $\{\xi_N \leq \xi_{N-1} \leq \cdots \leq \xi_1\}$ becomes the set $\{\zeta_{N/2} \leq \cdots \leq \zeta_1\} \cup \{0 \leq x_1\} \cup \{0 \leq x_j \leq \zeta_{j-1} - \zeta_j, j = 2, \cdots, N/2\}$. We will denote by $\mathbb{R}^{N/2}_+(\zeta)$ the set $\{0 \leq x_1\} \cup \{0 \leq x_j \leq \zeta_{j-1} - \zeta_j, j = 2, \cdots, N/2\}$. In the below, we use the notation $d\zeta = \prod_{j=1}^{N/2} d\zeta_j$ and $dx = \prod_{j=1}^{N/2} dx_j$, and also $w(\zeta) = \prod_{j=1}^{N/2} w(\zeta_j)$. Then the integral on the left hand side of (1.8) without the limit $A \to +\infty$ becomes

$$(*) := Z^{-1}_{N,A;w} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{\mathbb{R}^{N/2}_+(\zeta)} f p(x, \zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j + \zeta_j) dx_j, \tag{2.1}$$

where

$$\begin{aligned} p(x, \zeta) = \prod_{i=1}^{N/2} x_i \prod_{1 \leq i < j \leq N/2} (x_i - x_j + \zeta_i - \zeta_j) \\ \times \prod_{1 \leq i < j \leq N/2} (x_i + \zeta_i - \zeta_j) \prod_{1 \leq i < j \leq N/2} (\zeta_i - \zeta_j - x_j) \prod_{1 \leq i < j \leq N/2} (\zeta_i - \zeta_j) \end{aligned} \tag{2.2}$$

is a positive polynomial, and also each factor is non-negative.

Let $0 < \epsilon < 1$ be any fixed number. Let $B^{N/2}_\epsilon := \{0 \leq x_j \leq \epsilon, j = 1, \cdots, N/2\}$. We divide the integral with respect to x into two regions : (1) $X^\epsilon_1 := \mathbb{R}^{N/2}_+(\zeta) \cap B^{N/2}_\epsilon$ and (2) $X^\epsilon_2 := \mathbb{R}^{N/2}_+(\zeta) \setminus X^\epsilon_1$.

We first show that $Z^{-1}_{N,A;w}$ has a polynomial growth as $A \to \infty$.

Lemma 2.1. *We have*

$$\lim_{A \to \infty} A^N Z_{N,A;w} = Z_{N,\infty;w}. \tag{2.3}$$

where $Z_{N,\infty;w}$ is defined in (1.9).

Proof. For $x \in X^\epsilon_2$, since $e^{-Ax_j} \leq 1$ and there is at least one x_j satisfying $x_j > \epsilon$, we have

$$\begin{aligned} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{X^\epsilon_2} p(x, \zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j + \zeta_j) dx_j \\ \leq e^{-A\epsilon} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{X^\epsilon_2} p(x, \zeta) \prod_{j=1}^{N/2} w(x_j + \zeta_j) dx_j \leq e^{-A\epsilon} Z_{N,0;w}. \end{aligned} \tag{2.4}$$

For $x \in X_1^\epsilon$, since $\|V'\|_{L^\infty(\mathbb{R})} \leq C_0$, we have $w(\zeta_j)e^{-\epsilon C_0} \leq w(x_j + \zeta_j) \leq w(\zeta_j)e^{\epsilon C_0}$ for all $1 \leq j \leq N/2$. Thus

$$Z_{N,A;w} \leq e^{\frac{1}{2}\epsilon N C_0} \int_{\mathbb{R}^{N/2}_{ord}} \prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j \int_{X_1^\epsilon} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} dx_j + e^{-A\epsilon} Z_{N,0;w}, \tag{2.5}$$

and

$$Z_{N,A;w} \geq e^{-\frac{1}{2}\epsilon N C_0} \int_{\mathbb{R}^{N/2}_{ord}} \prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j \int_{X_1^\epsilon} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} dx_j. \tag{2.6}$$

In (2.2), for $x \in X_1^\epsilon$ and $\zeta \in \mathbb{R}^{N/2}_{ord}$, we have

$$\begin{aligned} p(x,\zeta) &\leq \prod_{i=1}^{N/2} x_i \prod_{i<j} (\epsilon + \zeta_i - \zeta_j) \prod_{i<j} (\epsilon + \zeta_i - \zeta_j) \prod_{i<j} (\zeta_i - \zeta_j) \prod_{i<j} (\zeta_i - \zeta_j) \\ &\leq (\Delta_{N/2}(\zeta)^4 + \epsilon Q(\zeta)) \prod_{k=1}^{N/2} x_k, \end{aligned} \tag{2.7}$$

where Δ is the Vandermonde product (1.1), and $Q(\zeta)$ is a positive polynomial. Thus using

$$\int_0^\infty x e^{-Ax} dx = A^{-2}, \tag{2.8}$$

(2.5) is less than or equal to

$$e^{\frac{1}{2}\epsilon N C_0} \frac{1}{A^N} \int_{\mathbb{R}^{N/2}_{ord}} (\Delta_{N/2}(\zeta)^4 + \epsilon Q(\zeta)) \prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j + e^{-A\epsilon} Z_{N,0;w}. \tag{2.9}$$

But

$$\int_{\mathbb{R}^{N/2}_{ord}} \Delta_{N/2}(\zeta)^4 \prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j = Z_{N,\infty;w}, \tag{2.10}$$

and hence

$$Z_{N,A;w} \leq \frac{1}{A^N} e^{\frac{1}{2}\epsilon N C_0} (Z_{N,\infty;w} + \epsilon C_1) + e^{-A\epsilon} Z_{N,0;\infty}, \tag{2.11}$$

for some constant $C_1 > 0$. This implies

$$\limsup_{A\to\infty} A^N Z_{N,A;w} \leq Z_{N,\infty;w}. \tag{2.12}$$

Similarly to (2.7), we have

$$p(x,\zeta) \geq (\Delta_{N/2}(\zeta)^4 - \epsilon R(\zeta)) \prod_{k=1}^{N/2} x_k \tag{2.13}$$

for some positive polynomial $R(\zeta)$. Hence from (2.6),

$$Z_{N,A;w} \geq e^{-\frac{1}{2}\epsilon N C_0} \int_{\mathbb{R}^{N/2}_{ord}} (\Delta_{N/2}(\zeta)^4 - \epsilon R(\zeta)) \prod_{j=1}^{N/2} w(\zeta_j)^2 d\zeta_j \int_{X_1^\epsilon} \prod_{j=1}^{N/2} x_j e^{-Ax_j} dx_j. \tag{2.14}$$

Since $X_1^\epsilon = \mathbb{R}_+^{N/2}(\zeta) \cap B_\epsilon^{N/2}$, we have

$$\int_{X_1^\epsilon} \prod_{j=1}^{N/2} x_j e^{-Ax_j} dx_j = \frac{1}{A^N} \prod_{j=1}^{N/2} (1 - (A+1) e^{-A \min(\epsilon, \zeta_{j-1} - \zeta_j)}), \tag{2.15}$$

where $\zeta_0 := +\infty$. We note that (2.15) is bounded below by $A^{-N}(1-(A+1)e^{-A\epsilon})^N$, and above by A^{-N}, if we take A large enough. Hence by considering (2.14) as sum of two integrals, one involving $\Delta^4_{N/2}$ and the other involving R, we find that

$$Z_{N,A;w} \geq e^{-\frac{1}{2}\epsilon N C_0} \frac{Z_{N,\infty;w}}{A^N} (1-(A+1)e^{-A\epsilon})^N - \epsilon e^{-\frac{1}{2}\epsilon N C_0} \frac{C_2}{A^N} \tag{2.16}$$

for some constant $C_2 > 0$. Therefore we obtain

$$Z_{N,\infty;w} \leq \liminf_{A \to \infty} A^N Z_{N,A;w}. \tag{2.17}$$

$\square$

Proof of (1.8)

Fix $0 < \epsilon < 1$. Then there is a $\delta > 0$ such that for $x \in X_1^\delta$, we have

$$\left| f(\zeta_1 + x_1, \zeta_1, \zeta_2 + x_2, \zeta_2, \cdots) - g(\zeta_1, \zeta_2, \cdots) \right| < \epsilon. \tag{2.18}$$

We write (2.1) as

$$(*) = (**) + (*1) + (*2) + (*3), \tag{2.19}$$

where

$$(**) = Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} g w(\zeta) d\zeta \int_{\mathbb{R}_+^{N/2}(\zeta)} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j + \zeta_j) dx_j, \tag{2.20}$$

and

$$\begin{aligned}
(*1) &= Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{X_2^\delta} f p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j + \zeta_j) dx_j, &(2.21)\\
(*2) &= -Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} g w(\zeta) d\zeta \int_{X_2^\delta} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j + \zeta_j) dx_j, &(2.22)\\
(*3) &= Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{X_1^\delta} (f-g) p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j + \zeta_j) dx_j. &(2.23)
\end{aligned}$$

As in (2.4), we have

$$|(*1)| \leq \|f\|_{L^\infty} e^{-A\delta} \frac{Z_{N,0;w}}{Z_{N,A;w}}, \qquad |(*2)| \leq \|g\|_{L^\infty} e^{-A\delta} \frac{Z_{N,0;w}}{Z_{N,A;w}}. \tag{2.24}$$

On the other hand, from (2.18),

$$|(*3)| \leq \epsilon Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{X_1^\delta} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j+\zeta_j) dx_j \leq \epsilon. \tag{2.25}$$

Thus using Lemma 2.1, $\limsup_{A\to\infty} |(*1)+(*2)+(*3)| \leq \epsilon$, but $\epsilon > 0$ is arbitrarily small, hence we have

$$\lim_{A\to\infty} |(*) - (**)| = 0. \tag{2.26}$$

Now the only remaining thing is to show that as $A \to \infty$.

$$(**) \to Z_{N,\infty;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} g \prod_{1\leq i<j\leq N/2} (\zeta_i - \zeta_j)^4 \prod_{j=1}^{N/2} (w(\zeta_j))^2 d\zeta_j. \tag{2.27}$$

Fix $0 < \epsilon' < 1$. We write

$$(**) = (**1) + (**2) + (**3), \tag{2.28}$$

where

$$\begin{aligned}
(**1) &= Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} g w(\zeta) d\zeta \int_{X_2^{\epsilon'}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j+\zeta_j) dx_j, &(2.29)\\
(**2) &= Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} g w(\zeta) d\zeta \int_{X_1^{\epsilon'}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} (w(x_j+\zeta_j) - w(\zeta_j)) dx_j, &(2.30)\\
(**3) &= Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} g w(\zeta)^2 d\zeta \int_{X_1^{\epsilon'}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} dx_j. &(2.31)
\end{aligned}$$

As in (2.4), we find

$$|(**1)| \leq \|g\|_{L^\infty} Z_{N,A;w}^{-1} e^{-A\epsilon'} Z_{N,0;w} \to 0, \tag{2.32}$$

as $A \to \infty$, using Lemma 2.1. For the estimation of $(**2)$, we use $|w(x_j+\zeta_j) - w(x_j)| \leq D(\epsilon') w(x_j+\zeta_j)$ for $x_j \in X_1^{\epsilon'}$, where $D(\epsilon') := \max(|1-e^{\epsilon' C_0}|, |1-e^{-\epsilon' C_0}|)$, and find that

$$|(**2)| \leq D(\epsilon') \|g\|_{L^\infty} Z_{N,A;w}^{-1} \int_{\mathbb{R}^{N/2}_{ord}} w(\zeta) d\zeta \int_{X_1^{\epsilon'}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j+\zeta_j) dx_j \leq D(\epsilon') \|g\|_{L^\infty} \tag{2.33}$$

where the second inequality is obtained by replacing the region $X_1^{\epsilon'}$ by $\mathbb{R}_+^{N/2}(\zeta)$ and noting the the total integral is 1 by (2.1). For $(**3)$, we note that (2.7), (2.8) and Lemma 2.1 yield

$$\begin{aligned}
0 &\leq Z_{N,A;w}^{-1} \int_{X_1^{\epsilon'}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} dx_j \leq (\Delta_{N/2}(\zeta)^4 + \epsilon' Q(\zeta)) Z_{N,A;w}^{-1} \int_{X_1^{\epsilon'}} \prod_{j=1}^{N/2} x_j e^{-Ax_j} dx_j \\
&\leq (\Delta_{N/4}(\zeta)^4 + \epsilon' Q(\zeta)) Z_{N,A;w}^{-1} A^{-N} \leq (\Delta_{N/4}(\zeta)^4 + \epsilon' Q(\zeta)) c Z_{N,\infty;w}^{-1}.
\end{aligned} \tag{2.34}$$

for some constant $c > 1$. But $\Delta_{N/4}(\zeta)^4 + \epsilon' Q(\zeta)$ is integrable with respect to the measure $w(\zeta)^2 d\zeta$ on $\mathbb{R}^{N/2}_{ord}$, and hence we can use the Lebesgue dominated convergence theorem. But an argument similar to (2.7)-(2.17) shows that for each $\zeta \in \mathbb{R}^{N/2}_{ord}$,

$$\lim_{A\to\infty} Z^{-1}_{N,A;w} \int_{X_1^{\epsilon'}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} dx_j = Z^{-1}_{N,\infty;w} \Delta_{N/4}(\zeta)^4, \tag{2.35}$$

and we find

$$\lim_{A\to\infty} (**3) = Z^{-1}_{N,\infty;w} \int_{\mathbb{R}^{N/2}_{ord}} g \prod_{1\le i<j\le N/2} (\zeta_i - \zeta_j)^4 \prod_{j=1}^{N/2} (w(\zeta_j))^2 d\zeta_j. \tag{2.36}$$

Thus the estimates (2.32), (2.33), (2.36), together with the fact that $D(\epsilon') \to 0$ as $\epsilon \to 0$, yield (2.27), and we prove the Proposition 1.1.

Proof of (1.10)

In the analysis above, the fact that f is uniformly continuous is used only for the estimation of $(*3)$. Now when $f = 1_{\xi_N \le \cdots \le \xi_1 \le t} = 1_{0\le x_1\le t-\zeta_1, \zeta_1<t}$ and $g = 1_{\zeta_1 \le t}$, $(*3)$ becomes

$$\begin{aligned}(*3) =& Z^{-1}_{N,A;w} \int_{\mathbb{R}^{N/2}_{ord}\cap\{\zeta_1<t\}} w(\zeta)d\zeta \int_{X_1^\delta \cap\{0\le x_1\le t-\zeta_1\}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j+\zeta_j)dx_j \\ &- Z^{-1}_{N,A;w} \int_{\mathbb{R}^{N/2}_{ord}\cap\{\zeta_1<t\}} w(\zeta)d\zeta \int_{X_1^\delta} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j+\zeta_j)dx_j\end{aligned} \tag{2.37}$$

where now $0 < \delta < 1$ can be taken to be an arbitrary fixed constant. When $t - \zeta_1 > \delta$, $X_1^\delta \cap \{0 \le x_1 \le t - \zeta_1\} = X_1^\delta$, and the above two integrals in x are the same, and hence we have

$$|(*3)| \le Z^{-1}_{N,A;w} \int_{\mathbb{R}^{N/2}_{ord}\cap\{t-\delta\le\zeta_1<t\}} w(\zeta)d\zeta \int_{X_1^\delta\cap\{t-\zeta_1<x_1\le\delta\}} p(x,\zeta) \prod_{j=1}^{N/2} e^{-Ax_j} w(x_j+\zeta_j)dx_j. \tag{2.38}$$

As in Lemma 2.1, this can be estimated as

$$|(*3)| \le Z^{-1}_{N,A;w} A^N e^{\frac{1}{2}\delta N C_0} \int_{\mathbb{R}^{N/2}_{ord}\cap\{t-\delta\le\zeta_1<t\}} (\Delta_{N/4}(\zeta)^4 + \delta Q(\zeta)) w(\zeta)^2 d\zeta. \tag{2.39}$$

But $Z^{-1}_{N,A;w} A^N$ is bounded as $A \to \infty$, and the integral vanishes when we take $\delta \to 0$, and hence we obtain $(*3) \to 0$ as $A \to \infty$. The rest of analysis is the same as for the proof of (1.8).

Remark

In addition to the change $\beta \to 4\beta$, we can also obtain the transition $\beta \to k^2\beta$ for each $k \in \mathbb{N}$. Let

$$\eta_3(\xi; A) := e^{-A(\xi_1+\xi_2-2\xi_3+\xi_4+\xi_5-2\xi_6+\cdots)}, \tag{2.40}$$

and similarly we set $\eta_k(\xi; A)$ with the term $\xi_1 + \xi_2 + \cdots + \xi_{k-1} - (k-1)\xi_k + \xi_{k+1} - \cdots$. Then when N is a multiple of k,

$$\begin{aligned}\lim_{A\to+\infty} Z^{-1}_{N,A;w;\beta;k} \int_{\mathbb{R}^N_{ord}} f(\xi)\Delta_N(\xi)^\beta \eta_k(\xi; A) \prod_{j=1}^{N} w(\xi_j) d\xi_j \\ = Z^{-1}_{N,\infty;w;\beta;k} \int_{\mathbb{R}^{N/k}_{ord}} f_k(\zeta)\Delta_{N/k}(\zeta)^{k^2\beta} \prod_{j=1}^{N/k} (w(\zeta_j))^k d\zeta_j.\end{aligned} \tag{2.41}$$

where f_k is obtained from f by setting the first k variables equal, and the next k variables equal, and so on. There are other possible choices of η_k. For instance,

$$\eta_k(\xi; A) = e^{-A(\xi_1 - \xi_k + \xi_{k+1} - \xi_{2k} + \cdots)} \tag{2.42}$$

would again yield (2.41).

3 First formula for geometric percolation : multi-sum expression

For $0 < q < 1$, let $g(q)$ denote the geometric random variable with parameter q : for $k = 0, 1, 2, \cdots$, $\mathbb{P}(g(q) = k) = (1-q)q^k$. We consider a last passage percolation model with geometric random variables analogous to the percolation model with exponential random variables considered in the Introduction. Namely, to each site $(i, j) \in \mathbb{Z}_+^2$ the random variable $X(i, j)$ is attached where

$$X(i, j) \sim g(q), \qquad i < j, \tag{3.1}$$

$$X(i, i) \sim g(\alpha\sqrt{q}), \tag{3.2}$$

$$X(j, i) = X(j, i), \qquad i < j. \tag{3.3}$$

Here $q \in (0, 1)$ and $\alpha \in (0, 1/\sqrt{q})$ are fixed numbers. As before, the random variables are independent except the symmetry condition $X(i, j) = X(j, i)$. As in the Introduction, $\Pi(N)$ denotes the set of up/right paths from $(1, 1)$ to (N, N). We define

$$G^{\boxslash}(N; \alpha) = \max\big\{ \sum_{(i,j)\in\pi} X(i, j) : \pi \in \Pi(N) \big\}. \tag{3.4}$$

Since a proper limit of the geometric random variable becomes the exponential random variable, we find that the random variable $H^{\boxslash}(N; \rho)$ is a limit of $G^{\boxslash}(N.\alpha)$.

Lemma 3.1. *We have*

$$\mathbb{P}(H^{\boxslash}(N; \rho) \le x) = \lim_{L\to\infty} \mathbb{P}(G^{\boxslash}(N; \alpha) \le xL), \tag{3.5}$$

where we set

$$\sqrt{q} = 1 - \frac{1}{2L}, \qquad \alpha = 1 - \Big(\frac{1}{\rho} - \frac{1}{2}\Big)\frac{1}{L}. \tag{3.6}$$

Proof. It is direct to check that with (3.6),

$$e(1) = \lim_{L\to\infty} \frac{g(q)}{L}, \qquad e(\rho) = \lim_{L\to\infty} \frac{g(\alpha\sqrt{q})}{L}, \tag{3.7}$$

in distribution. Thus under this limit, the last passage percolation model with geometric random variables becomes the last passage percolation model with exponential random variables. It is also direct to check that $G^{\boxslash}(N,\alpha)/L \to H^{\boxslash}(N,\rho)$ in distribution. $\square$

Now the key thing is that there are two different formulas for $\mathbb{P}(G^{\boxslash}(N,\alpha) \le n)$. Thus by taking the exponential limit $L \to \infty$ with (3.6), we would obtain two different formulas for $\mathbb{P}(H^{\boxslash}(N,\rho) \le x)$. It will turn out that one of the limiting formula is the multi-integral formula given in the right-hand-side of (1.16) which represents the probability distribution for the rightmost 'particle' in the interpolating ensemble, and the other is the Painlevé V expression, Theorem 1.3. We present the first formula for $\mathbb{P}(G^{\boxslash}(N,\alpha) \le n)$ in this section. The second formula will be considered in the subsequent sections.

The following lemma is modeled on the paper [19] in which a similar result for the unsymmetrized case (at each site (i,j), the geometric random variables $X(i,j)$ are independent and identically distributed without the symmetry condition $X(i,j) = X(j,i)$).

Lemma 3.2. *Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We have*

$$\mathbb{P}(G^{\boxslash}(N;\alpha) \le n) = Z_1(N,\alpha)^{-1} \sum_{\substack{0\le h_N<\cdots<h_1\le n+N-1 \\ h_j\in\mathbb{N}_0}} \prod_{1\le i<j\le N} (h_i - h_j) \prod_{i=1}^{N} q^{h_i/2} \alpha^{-(-1)^j h_j}, \tag{3.8}$$

where the normalization constant is

$$Z_1(N,\alpha) = (1-\alpha\sqrt{q})^{-N} (1-q)^{-N(N-1)/2} \alpha^{[N/2]} q^{N(N-1)/2} \prod_{j=0}^{N-1} j!. \tag{3.9}$$

Remark. The result for the special case when $\alpha = 1$ is stated in Remark 5.2 [19].

Proof. (cf. Section 2.1, [19] and Proof of Theorem 7.1, [5]) Let $\lambda = (\lambda_1, \cdots, \lambda_N)$ is a Young diagram : a sequence of integers, $\lambda_1 \ge \cdots \ge \lambda_N \ge 0$. The number $d_\lambda(N)$ of semistandard Young tableaux (SSYT) of shape λ with elements taken from $\{1, \cdots, N\}$ is equal to (see e.g., [25])

$$d_\lambda(N) = \prod_{1\le i<j\le N} \frac{\lambda_i - \lambda_j + j - i}{j-i} = \prod_{j=0}^{N-1} \frac{1}{j!} \prod_{1\le i<j\le N} (h_i - h_j), \qquad h_j := \lambda_j + N - j. \tag{3.10}$$

Let $D(N) = \sum_{j=1}^N X(j,j)$ and $OD(N) = \sum_{1\le i<j\le N} X(i,j)$. Then

$$\begin{aligned} &\mathbb{P}(G^{\boxslash}(N;\alpha) \le n) \\ &= \sum_{k,m\ge0} \mathbb{P}(G^{\boxslash}(N;\alpha) \le n | D(N) = m, OD(N) = k)\, \mathbb{P}(D(N) = m, OD(N) = k) \end{aligned} \tag{3.11}$$

Let $I_N^{m,k}$ be the set of $N \times N$ symmetric matrices with non-negative integer-valued entries such that the sum of diagonal entries is m and the sum of upper-triangular entries is equal to k. For $A = (a_{ij})_{1\le i,j\le N} \in I_N^{m,k}$, the probability that the random matrix $(X(i,j))_{1\le i,j\le N}$ becomes A is given by

$$\prod_j (1-\alpha\sqrt{q})(\alpha\sqrt{q})^{a_{jj}} \prod_{i<j} (1-q)q^{a_{ij}} = (1-\alpha\sqrt{q})^N (1-q)^{N(N-1)/2} \alpha^m q^{(2k+m)/2}, \tag{3.12}$$

which is a value independent of the choice of $A \in I_N^{m,k}$. Hence we have

$$\mathbb{P}(D(N) = m, OD(N) = k) = \# I_N^{k,m} (1-\alpha\sqrt{q})^N (1-q)^{N(N-1)/2} \alpha^m q^{(2k+m)/2}. \tag{3.13}$$

Also the conditional probability $\mathbb{P}(\cdot | D(N) = m, OD(N) = k)$ is the uniform distribution on $I_N^{k,m}$. Now we use a version of the Robinson-Schensted-Knuth correspondence which yields a bijection between $I_N^{k,m}$ and the set of semistandard Young tableaux with shapes $\lambda \vdash 2k+m$ such that $f(\lambda) = m$, where

$$f(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots \tag{3.14}$$

denotes the number of odd columns of λ. Moreover, under the Robinson-Schensted-Knuth correspondence, the length of the longest up/right path $G^{\boxslash}$ is equal to the first row λ_1. Therefore, we have

$$\mathbb{P}(G^{\boxslash}(N;\alpha) \le n | D(N) = m, OD(N) = k) = \frac{1}{\# I_N^{k,m}} \sum_{\substack{\lambda \vdash 2k+m \\ f(\lambda)=m \\ \lambda_1 \le n}} d_\lambda(N). \tag{3.15}$$

Thus from (3.11), (3.13) and (3.15),

$$\mathbb{P}(G^{\boxslash}(N;\alpha) \le n) = (1-\alpha\sqrt{q})^N (1-q)^{N(N-1)/2} \sum_{\lambda_1 \le n} d_\lambda(N) \alpha^{f(\lambda)} q^{(\lambda_1+\lambda_2+\cdots)/2} \tag{3.16}$$

where the sum is over all Young diagrams λ satisfying $\lambda_1 \le n$. Note that if $\lambda = (\lambda_1, \cdots \lambda_k)$ with $\lambda_k > 0$ and $k > N$, we have $d_\lambda(N) = 0$, and hence the sum in (3.16) is over the Young diagrams $\lambda = (\lambda_1, \cdots, \lambda_N)$ satisfying $\lambda_1 \le n$. Now we use (3.10) and set $h_j = \lambda_n + N - j$, and the results (3.8) and (3.9) are obtained. $\square$

By taking the limit (3.5) of (3.8), we obtain Proposition 1.2, which was originally given in Remark 7.6.1, [5].

4 Second formula for geometric percolation : orthogonal polynomials

In this section, we present the second formula for the geometric percolation model introduced in Section 3. The following determinant formula is given in Theorem 7.1 of [5] for a more general model.

Lemma 4.1. *(Theorem 7.1, [5]) We have*

$$\mathbb{P}(G^{\boxslash}(N;\alpha) \le n) = Z_3(N,\alpha)^{-1}\, \mathbb{E}_{U\in O(n)} \det((1+\alpha U)(1+\sqrt{q}U)^N) \tag{4.1}$$

with

$$Z_3(N,\alpha) = (1-\alpha\sqrt{q})^{-N}(1-q)^{-N(N-1)/2}. \tag{4.2}$$

Proof. The random variable $G^{\boxslash}(N;\alpha)$ is same as $\ell^{\boxslash}_{W}(\overrightarrow{q};\alpha)$ in Theorem 7.1 of [5], where $W = \mathbb{Z}_+$ and $\overrightarrow{q} = (\sqrt{q},\cdots,\sqrt{q},0,0,\cdots)$ with $\sqrt{q}$'s occurring N times. (In Section 7 of [5], $\overrightarrow{q}$ is denoted by q, but we take this notation to avoid the confusion with the parameter $0<q<1$ of the geometric random variable.) By (7.32) of [5], we have

$$\mathbb{P}(G^{\boxslash}(N;\alpha)\le n) = \mathbb{P}(\ell^{\boxslash}_{\mathbb{Z}_+}(\overrightarrow{q};\alpha)\le n) = Z^{\boxslash}_{\mathbb{Z}_+}(\overrightarrow{q};\alpha)\sum_{\ell(\lambda)\le n}\alpha^{f(\lambda)}s_\lambda(/\overrightarrow{q}_{\mathbb{Z}_+}) \tag{4.3}$$

where $\ell(\lambda)$ is the number of parts of the Young diagram λ and $f(\lambda)$ is same as (3.14). Indeed this formula is, after the modification $\lambda\mapsto\lambda^t$, is equal to the formula (3.16) in Section 3. The normalization constant is by (7.11) of [5],

$$Z_3(N,\alpha) = Z^{\boxslash}_{\mathbb{Z}_+}(\overrightarrow{q};\alpha)^{-1} = (1-\alpha\sqrt{q})^{-N}(1-q)^{-N(N-1)/2}. \tag{4.4}$$

From (5.55) of [5] with the modification as in the paragraph preceding Theorem 7.1 of [5], the sum in (4.3) is equal to

$$\sum_{\ell(\lambda)\le n}\alpha^{f(\lambda)}s_\lambda(/\overrightarrow{q}_{\mathbb{Z}_+}) = \mathbb{E}_{U\in O(n)}\det((1+\alpha U)H(U;\overrightarrow{0}/\overrightarrow{q}_{\mathbb{Z}_+})). \tag{4.5}$$

But by (7.28) and (5.6) of [5], we have

$$H(U;\overrightarrow{0}/\overrightarrow{q}_{\mathbb{Z}_+}) = E(U;\overrightarrow{q}_{\mathbb{Z}_+}) = (1+\sqrt{q}U)^N, \tag{4.6}$$

and hence we obtain (4.1). $\square$

Let $O(n)_\pm$ denote the connected component of $O(n)$ with $\det(U)=\pm 1$, respectively. The authors in [5] expressed the expected value over the orthogonal group in (4.1) in terms of the related orthogonal polynomials. Set

$$\psi(z) = \psi(z;M,N) := (1+\sqrt{q}z)^N(1+\sqrt{q}z^{-1})^N. \tag{4.7}$$

Let $\pi_j(z)$ be the monic orthogonal polynomial of degree j with respect to the measure $\psi(z)dz/(2\pi iz)$ on the unit circle :

$$\int_{|z|=1}\pi_j(z)\overline{\pi_k(z)}\psi(z)\frac{dz}{2\pi iz} = N_j\delta_{jk}, \tag{4.8}$$

where N_j is the square of the norm of $\pi_j(z)$. We also set

$$\pi_j^* = z^j\pi_j(z^{-1}). \tag{4.9}$$

Remark. In general, π_j^* is defined by $z^j\overline{\pi_j}(z^{-1})$. But for the case at hand, all the coefficients of π_j are real, and hence taking the complex conjugate has no effect.

We also set

$$\phi_j(z) \quad = (1+\sqrt{q}z)^N \pi_j(z), \tag{4.10}$$

$$\phi_j^*(z) \quad = (1+\sqrt{q}z)^N \pi^*(z). \tag{4.11}$$

Note that a version of strong Szegö theorem yields that (see e.g., [18])

$$\lim_{n\to\infty} \mathbb{E}_{U\in O(n)_\pm} \det((1+\sqrt{q}U)^N) = (1-q)^{-N(N-1)/2}. \tag{4.12}$$

From Theorem 3.1 and Theorem 2.3 in [5], (4.1) has the following expression.

Lemma 4.2. *For $n \geq 1$,*

$$\mathbb{P}(G^{\boxslash}(N;\alpha) \leq 2n) = \frac{1}{2}\Big\{[\phi^*_{2n-1}(-\alpha) - \alpha\phi_{2n-1}(-\alpha)]\Delta_n^{--} + [\phi^*_{2n-1}(-\alpha) + \alpha\phi_{2n-1}(-\alpha)]\Delta_{n-1}^{++}\Big\} \tag{4.13}$$

$$\mathbb{P}(G^{\boxslash}(N;\alpha) \leq 2n+1) = \frac{1}{2}\Big\{[\phi^*_{2n}(-\alpha) + \alpha\phi_{2n}(-\alpha)]\Delta_n^{+-} + [\phi^*_{2n}(-\alpha) - \alpha\phi_{2n}(-\alpha)]\Delta_n^{-+}\Big\} \tag{4.14}$$

$$\tag{4.15}$$

where

$$\Delta_n^{--} = \prod_{j\geq n} N_{2j+2}^{-1}(1+\pi_{2j+2}(0)) \tag{4.16}$$

$$\Delta_n^{++} = \prod_{j\geq n} N_{2j+2}^{-1}(1-\pi_{2j+2}(0)) \tag{4.17}$$

$$\Delta_n^{+-} = \prod_{j\geq n} N_{2j+1}^{-1}(1-\pi_{2j+1}(0)) \tag{4.18}$$

$$\Delta_n^{-+} = \prod_{j\geq n} N_{2j+1}^{-1}(1+\pi_{2j+1}(0)). \tag{4.19}$$

We also have for $n \geq 1$,

$$\mathbb{P}(G^{\boxslash}(N;1) \leq 2n) = \Delta_n^{-+}$$

$$\mathbb{P}(G^{\boxslash}(N;1) \leq 2n+1) = \Delta_n^{++}$$

The main results, Theorem 1.3 will be obtained by analyzing the orthogonal polynomials π_k asymptotically. This asymptotic analysis will be carried out in Section 6. But we first need the following section which will be used for the analysis in Section 6.

5 Painlevé V

In this Section, we prove various properties of a Riemann-Hilbert problem for Painlevé V solution. These properties will be used in the next Section 6 for the analysis of orthogonal polynomials π_k, and also for the proof of Theorem 1.3. This section is, however, independent of other sections.

Fix $0 < a < 1$. Let $\Gamma_1 = \{w \in \mathbb{C} : |w-1| = a\}$ and $\Gamma_2 = \{w : |w+1| = a\}$. We orient the circle Γ_1 counter-clockwise, and orient the circle Γ_2 clockwise. Set $\Gamma = \Gamma_1 \cup \Gamma_2$. Let Ω_1, Ω_2 and Ω_0 be the open regions as indicated in Figure 1.

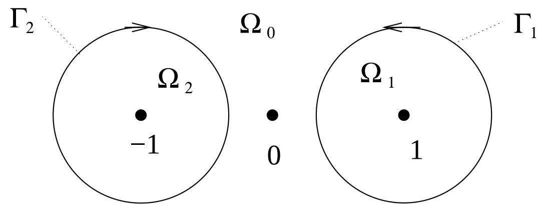

Figure 1: Contours Γ_1, Γ_2 and the regions $\Omega_1, \Omega_2, \Omega_0$

Let

$$\Phi(w) = \Phi(w; x, N) := \frac{(1+w)^N}{(1-w)^N} e^{-\frac{1}{2}xw}. \tag{5.1}$$

Define the 2×2 matrix $V(w) = V(w; x, N)$ on Γ by

$$\begin{cases} V(w) = \begin{pmatrix} 1 & -\Phi(w) \\ 0 & 1 \end{pmatrix}, & w \in \Gamma_1, \\ V(w) = \begin{pmatrix} 1 & 0 \\ \Phi(w)^{-1} & 1 \end{pmatrix}, & w \in \Gamma_2. \end{cases} \tag{5.2}$$

Let the 2×2 matrix $M(w) = M(w; x, N) = (M_{jk}(w))_{j,k=1,2}$ be the solution to the following Riemann-Hilbert problem (RHP) :

$$\begin{cases} M(w) \quad \text{is analytic in } w \in \mathbb{C} \setminus \Gamma, \\ M_+(w) = M_-(w)V(w), \qquad w \in \Gamma, \\ M(w) = I + O(w^{-1}), \qquad \text{as } w \to \infty. \end{cases} \tag{5.3}$$

The solution M shares the following properties.

Proposition 5.1. *Set* $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and set* $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. *We have the following properties :*

(i). There is a unique solution $M(w) = M(w; x, N)$ *to the RHP* (5.3) *for each* $x > 0$ *and* $N \in \mathbb{N}$, *and the solution* M *has the expansion*

$$M(w) = I + \frac{M_1}{w} + \frac{M_2}{w^2} + \cdots, \qquad w \to \infty. \tag{5.4}$$

(ii). $M(w) = \overline{M(\overline{w})}$, $M(w) = \sigma_1 M(-w) \sigma_1$, *and* $\det M(w) = 1$.

(iii). $M(w)$ *is real for* $w \in \mathbb{R}$, *and* M_1 *and* M_2 *have the form*

$$M_1 = \begin{pmatrix} -\alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \frac{1}{2}(\alpha^2 - \beta^2) & \gamma \\ \gamma & \frac{1}{2}(\alpha^2 - \beta^2) \end{pmatrix}. \tag{5.5}$$

Here α, β, γ *are real constants which depend on* x *and* N.

(iv). We have

$$\frac{\partial}{\partial x}M = \frac{1}{4}w[M,\sigma_3] + \frac{1}{2}\beta\sigma_1 M. \tag{5.6}$$

This implies, in particular,

$$\alpha' = \frac{1}{2}\beta^2, \qquad \beta' = \frac{1}{2}(\alpha\beta - \gamma), \tag{5.7}$$

where $'$ denotes the derivative with respect to x,

(v). We have

$$\frac{\partial}{\partial w}M = \frac{1}{4}x[M,\sigma_3] - \frac{N}{1-w^2}M\sigma_3 + \frac{N}{2}\frac{1}{1-w}AM - \frac{N}{2}\frac{1}{1+w}\sigma_1 A\sigma_1 M, \tag{5.8}$$

where

$$A = M(1)\sigma_3 M(1)^{-1} = \sigma_3 + \frac{1}{N}M_1 + \frac{1}{2N}x\beta\sigma_1(-I+M_1) + \frac{1}{2N}x\gamma\sigma_1\sigma_3. \tag{5.9}$$

(vi). Set $\theta(x) = \theta(x;N) = -\frac{1}{2}x\alpha(x)$. Then θ solves the Painlevé V equation

$$(x\theta'')^2 = (\theta - x\theta')(\theta - x\theta' + 4(\theta')^2 + 4N\theta'). \tag{5.10}$$

(vii). Fix $0 < \epsilon < 1/4$. For each fixed $N \in \mathbb{N}$, as $x \to +\infty$,

$$\beta(x) = -2(-1)^N L_{N-1}^{(1)}(x)e^{-\frac{1}{2}x} + O(e^{-(1-\epsilon)x}) = \frac{-2x^{N-1}}{(N-1)!}e^{-\frac{1}{2}x}(1+O(x^{-1})) \tag{5.11}$$

and

$$\alpha(x) = \int_\infty^x 2L_{N-1}^{(1)}(y))^2 e^{-y}dy + O(e^{-\frac{3}{2}(1-\epsilon)x}) = \frac{2x^{2N-2}}{((N-1)!)^2}e^{-x}(1+O(x^{-1})), \tag{5.12}$$

where $L_{N-1}^{(1)}(x)$ is the Laguerre polynomial of degree $N-1$ with parameter 1 ;

$$L_{N-1}^{(1)}(x) = \sum_{j=0}^{N-1}\binom{N}{j+1}\frac{(-x)^j}{j!} = \frac{1}{2\pi i}\int_{|z|=\frac{1}{2}x}\big(1+\frac{x}{z}\big)^{N-1}e^{-z}\big(1+\frac{z}{x}\big)\frac{dz}{z}. \tag{5.13}$$

Also if we take Γ_1 and Γ_2 to be the circles of radius ϵ, centered at 1 and -1, respectively, we have for each fixed $w \in \mathbb{C}\setminus(\Gamma_1\cup\Gamma_2)$, as $x \to +\infty$,

$$M(w) = I - \begin{pmatrix} 0 & \Lambda(w,x,N) \\ \Lambda(-w,x,N) & 0 \end{pmatrix} + O(e^{-(1-2\epsilon)x}), \tag{5.14}$$

where

$$\Lambda(w,x,N) := \frac{1}{2\pi i}\int_{\Gamma_1}\Phi(s)\frac{ds}{s-w}. \tag{5.15}$$

(viii). We have

$$M(0)=\frac{1}{2}\begin{pmatrix}\mathrm{E_N}(x)^{-2}+\mathrm{E_N}(x)^{2} & \mathrm{E_N}(x)^{-2}-\mathrm{E_N}(x)^{2}\\ \mathrm{E_N}(x)^{-2}-\mathrm{E_N}(x)^{2} & \mathrm{E_N}(x)^{-2}+\mathrm{E_N}(x)^{2}\end{pmatrix} \tag{5.16}$$

where $\mathrm{E_N}(x)$ *is defined in* (1.20) *with* α *in* (5.5).

Proof. (i) The proof of (i) is parallel to the analysis of Theorem 5.50 in [10]. We here only present an outline of the proof. Let C be the Cauchy operator given by

$$(Cf)(w)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(s)}{s-w}ds, \qquad f\in L^2(\Gamma). \tag{5.17}$$

Let C_+ be the limit from the positive side of the contour :

$$(C_+f)(w)=\lim_{w'\to w}(Cf)(w'), \qquad w' \text{ is in the } (+)\text{-side of } \Gamma. \tag{5.18}$$

Define the operator C_V acting on $L^2(\Gamma)$ by

$$(C_Vf)(w):=C_+(f(I-V^{-1})(w), \qquad f\in L^2(\Gamma). \tag{5.19}$$

From the general theory (see e.g. Appendix I, [9]), if the operator $1-C_V$ is invertible in $L^2(\Gamma)$, the function $M(w)$ defined by

$$M(w)=I+\frac{1}{2\pi i}\int_{\Gamma}\frac{(I+(1-C_V)^{-1}(C_VI))(s)(I-V^{-1})(s)}{s-w}ds \tag{5.20}$$

solves the RHP (5.3) in L^2 sense. But since V and V^{-1} are analytic on Γ, the L^2 solution is indeed the classical solution of (5.3) (see e.g. Proposition 5.80, [10]). Just as in Step 1 and 2 of the proof of Theorem 5.50, [10], $1-C_V$ is a Fredholm operator of index 0, because the contour Γ is compact and V is real analytic on Γ. Thus it is enough to show that $Ker(1-C_V)=\{0\}$ in order to prove that $1-C_V$ is invertible. Now suppose $(1-C_V)f=0$ for $f\in L^2(\Gamma)$. We will show that $f=0$, which will prove that $1-C_V$ is invertible. Set

$$n(w):=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(s)(I-V^{-1})(s)}{s-w}ds. \tag{5.21}$$

From the properties of the Cauchy operator, n satisfies

- $n(w)$ is analytic in $\mathbb{C}\setminus\Gamma$, and continuous up to the boundaries.
- $n_+(w)=n_-(w)V(w), \qquad w\in\Gamma.$
- $n(w)=O(\frac{1}{w})$ as $w\to\infty$.

Let Γ_0 be the imaginary axis, oriented from $i\infty$ to $-i\infty$. Define $N(w)$ by

$$\begin{cases} N(w)=n(w)\begin{pmatrix}1 & -\Phi(w)\\ 0 & 1\end{pmatrix}, & w\in\Omega_0, Re(w)>0,\\ N(w)=n(w)\begin{pmatrix}1 & 0\\ -\Phi(w)^{-1} & 1\end{pmatrix}, & w\in\Omega_0, Re(w)<0,\\ N(w)=n(w) & w\in\Omega_1\cup\Omega_2. \end{cases} \tag{5.22}$$

Then N satisfies

– $N(w)$ is analytic in $w \in \mathbb{C} \setminus \Gamma_0$.

– $N_+(w) = N_-(w)V_0(w)$ for $w \in \Gamma_0$, where $V_0(w) = \begin{pmatrix} 1 & -\Phi(w) \\ \Phi(w)^{-1} & 0 \end{pmatrix}$.

– $N(w) = O(\frac{1}{w} e^{-\frac{1}{2}x|Re(w)|})$ as $w \to \infty$.

Now consider

$$a(w) := N(w)\overline{N(-\overline{w})}^T. \tag{5.23}$$

Then $a(w)$ is holomorphic in $\mathbb{C} \setminus \Gamma_0$, and hence by Cauchy's theorem and the decay property as $w \to \infty$, we have

$$\int_\Gamma a_+(w)dw = 0. \tag{5.24}$$

But from the jump condition of N,

$$a_+(w) = N_+(w)\overline{N_-(-\overline{w})}^T = N_+(w)\overline{N_-(w)}^T = N_+(w)(\overline{V_0(w)}^{-1})^T \overline{N_+(w)}^T \tag{5.25}$$

for $w \in \Gamma$. Thus using the property $\Phi(w)^{-1} = \overline{\Phi(w)}$ for $w \in \Gamma_0$,

$$\begin{aligned} 0 &= \int_\Gamma (a_+(w) + \overline{a_+(w)}^T)dw = \int_\Gamma N_+(w)\big((\overline{V_0(w)}^{-1})^T + V_0(w)^{-1}\big)\overline{N_+(w)}^T dw \\ &= \int_\Gamma N_+(w) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \overline{N_+(w)}^T dw. \end{aligned} \tag{5.26}$$

This implies that $(N_{12})_+(w) = (N_{22})_+(w) = 0$ for almost every $w \in \Gamma$, and hence by Cauchy's theorem, we have

$$N_{12}(w) = N_{22}(w) = 0, \qquad Re(w) > 0. \tag{5.27}$$

This in turn implies, from the jump condition, that

$$N_{11}(w) = N_{21}(w) = 0, \qquad Re(w) < 0. \tag{5.28}$$

On the other hand, from the jump condition again,

$$(N_{11})_+(w) = (N_{12})_-(w)e^{\frac{1}{2}xw}\left(\frac{1-w}{1+w}\right)^N, \qquad w \in i\mathbb{R}. \tag{5.29}$$

Set

$$h(w) := \begin{cases} (1+w)^N N_{11}(w), & Re(w) > 0, \\ (1-w)^N e^{\frac{1}{2}xw} N_{12}(w), & Re(w) < 0. \end{cases} \tag{5.30}$$

Then h is entire, and $h(w)w^{-N-1} \to 0$ as $w \to \infty$ since $N(w)$ is bounded for $w \in \mathbb{C}$. Thus by the Liouville's theorem, $h(w)$ is a polynomial of degree at most N. But then for $Re(w) < 0$, $N_{12}(w) = h(w)(1-w)^N e^{-\frac{1}{2}xw}$ blows up as $w \to \infty$ unless $h = 0$ identically. Therefore we obtain

$$\begin{aligned} N_{12}(w) = 0, \qquad Re(w) < 0, \\ N_{11}(w) = 0, \qquad Re(w) > 0. \end{aligned} \tag{5.31}$$

By a similar argument, we obtain

$$\begin{aligned} N_{21}(w) &= 0, \qquad Re(w) > 0, \\ N_{22}(w) &= 0, \qquad Re(w) < 0. \end{aligned} \tag{5.32}$$

Therefore we have $N(w) = 0$ and hence $n(w) = 0$ for all $w \in \mathbb{C}$. Since by (5.19) and (5.21), $n_+ = C_+(f(I - V^{-1})) = C_V(f) = f$, we obtain that $Ker(1 - C_V)$ has dimension 0. The uniqueness of the solution is standard. The expansion (5.4) follows from (5.21).

(ii) The jump matrix V have the properties, $V(w) = \overline{V(\overline{w})}$, $V(w) = \sigma_1 V(-w)^{-1}\sigma_1$ and $\det V(w) = 1$ for $w \in \Gamma$. These properties, together with the uniqueness of the solution to the RHP, imply the results.

(iii) The realness of $M(w)$ for $w \in \mathbb{R}$ follows from $M(w) = \overline{M(\overline{w})}$, which also implies that M_1 and M_2 are real. By expanding the both sides of $M(w) = \sigma_1 M(-w)\sigma_1$ as $w \to \infty$, we have $M_1 = -\sigma_1 M_1 \sigma_1$ and $M_2 = \sigma_1 M_2 \sigma_1$, and hence

$$M_1 = \begin{pmatrix} -\alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad M_2 = \begin{pmatrix} (M_2)_{11} & \gamma \\ \gamma & (M_2)_{11} \end{pmatrix}, \tag{5.33}$$

for some $\alpha, \beta, \gamma, (M_2)_{11}$. Also since $0 = \log\det M = tr \log M = tr \log(I + \frac{M_1}{w} + \frac{M_2}{w^2} + \cdots)$, as $w \to \infty$, we have $tr(M_1) = 0$ and $tr(M_2 - \frac{1}{2}M_1^2) = 0$. The second identity implies that $(M_2)_{11} = \frac{1}{2}(\alpha^2 - \beta^2)$ and we obtain the results.

(iv) Set $f := Me^{-\frac{1}{4}xw\sigma_3}$. Then

$$\begin{cases} f \quad \text{is analytic in } \mathbb{C} \setminus \Gamma, \\ f_+ = f_- v_f(w) \qquad \text{for } w \in \Gamma, \\ fe^{\frac{1}{4}xw\sigma_3} \to I \qquad \text{as } w \to \infty. \end{cases} \tag{5.34}$$

where

$$\begin{cases} v_f(w) = \begin{pmatrix} 1 & -(\frac{1+w}{1-w})^N \\ 0 & 1 \end{pmatrix} \qquad w \in \Gamma_1, \\ v_f(w) = \begin{pmatrix} 1 & 0 \\ (\frac{1-w}{1+w})^N & 1 \end{pmatrix} \qquad w \in \Gamma_2. \end{cases} \tag{5.35}$$

Note that the jump matrix does not depend on x. Thus f' and f satisfy the same jump condition, and hence $f'f^{-1}$ is a entire function, where $f' = \frac{\partial}{\partial x} f$. On the other hand, from the condition $fe^{\frac{1}{4}xw\sigma_3} \to I$ as $w \to \infty$, we obtain $f'f^{-1} + \frac{1}{4}wf\sigma_3 f^{-1} \to 0$ as $w \to \infty$. But since $M = I + \frac{M_1}{w} + O(w^{-2})$ as $w \to \infty$, we have $\frac{1}{4}wf\sigma_3 f^{-1} = \frac{1}{4}w\sigma_3 + \frac{1}{4}[M_1, \sigma_3] + O(w^{-1})$. Thus by Liouville's theorem, $f'f^{-1} = -\frac{1}{4}w\sigma_3 - \frac{1}{4}[M_1, \sigma_3]$, which is equivalent to

$$M' = \frac{1}{4}w[M, \sigma_3] + \frac{1}{2}\beta\sigma_1 M, \tag{5.36}$$

as desired.

By taking the limit $x \to \infty$ in (5.6) and collecting the terms of order $O(w^{-1})$, we have

$$M_1' = \frac{1}{4}[M_2, \sigma_3] + \frac{1}{2}\beta\sigma_1 M_1. \tag{5.37}$$

This implies (5.7).

(v) Set $h = Me^{-\frac{1}{4}xw\sigma_3}\left(\frac{1+w}{1-w}\right)^{\frac{1}{2}N\sigma_3}$, where $\left(\frac{1+w}{1-w}\right)^{\frac{1}{2}}$ is defined to be analytic in $\mathbb{C} \setminus [-1,1]$ with the condition that it becomes 1 as $w = iy \to 0$ satisfying $y > 0$. Then

$$\begin{cases} h \quad \text{is analytic in } \mathbb{C} \setminus ([-1,1] \cup \Gamma), \\ h_+ = h_- v_h, \qquad w \in \Gamma, \\ h_+ = (-1)^N h_-, \qquad w \in (-1,1), \\ he^{\frac{1}{4}xw\sigma_3}\left(\frac{1+w}{1-w}\right)^{-\frac{1}{2}N\sigma_3} \to I \qquad \text{as } w \to \infty, \end{cases} \tag{5.38}$$

where the interval $[-1,1]$ is oriented from the right to the left, and the jump matrix v_h is given by

$$\begin{cases} v_h = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad w \in \Gamma_1, \\ v_h = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad w \in \Gamma_2. \end{cases} \tag{5.39}$$

Then

$$\dot{h}h^{-1} = \dot{M}M^{-1} - \frac{1}{4}xM\sigma_3 M^{-1} + \frac{N}{1-w^2}M\sigma_3 M^{-1} \tag{5.40}$$

is analytic in $\mathbb{C} \setminus \{-1,1\}$ with simple poles at -1 and 1, where $\dot{h}$ denotes the derivative with respect to w. Thus, since $\dot{h}h^{-1} \to -\frac{1}{4}x\sigma_3$ as $w \to \infty$, we find that

$$\dot{h}h^{-1} = -\frac{1}{4}x\sigma_3 + \frac{A_0}{1-w} + \frac{B_0}{1+w} \tag{5.41}$$

with some constant matrices A_0 and B_0. By taking the limits of (5.40) as $w \to 1, -1$, we obtain $A_0 = \frac{N}{2}M(1)\sigma_3 M(1)^{-1}$ and $B_0 = \frac{N}{2}M(-1)\sigma_3 M(-1)^{-1}$. Since $M(-1) = \sigma_1 M(1)\sigma_1$ from (ii), we have $B_0 = -\sigma_1 A_0 \sigma_1$.

By combining (5.40) and (5.41), we obtain (5.8) with constant matrix $A = M(1)\sigma_3 M(1)^{-1}$. Now we take the limit $w \to \infty$ to (5.8). By collecting the terms of order $O(w^{-1})$ and noting that $[M_1, \sigma_3] = -2\beta\sigma_1$, we have

$$A + \sigma_1 A \sigma_1 = -\frac{1}{N}x\beta\sigma_1, \tag{5.42}$$

and by collecting the terms of order $O(w^{-2})$ and noting that $[M_2, \sigma_3] = 2\gamma\sigma_1\sigma_3$, we obtain

$$A - \sigma_1 A \sigma_1 = 2\sigma_3 + \frac{2}{N}M_1 + \frac{1}{N}x\beta\sigma_1 M_1 + \frac{1}{N}x\gamma\sigma_1\sigma_3. \tag{5.43}$$

These yield the second formula (5.9) for A.

(vi) From the second formula (5.9) of A, we have

$$A = \begin{pmatrix} 1 - \frac{1}{N}\alpha - \frac{1}{2N}x\beta^2 & \frac{1}{N} - \frac{1}{2N}x\beta(1-\alpha) - \frac{1}{2N}x\gamma \\ -\frac{1}{N} - \frac{1}{2N}x\beta(1+\alpha) + \frac{1}{2N}x\gamma & -1 + \frac{1}{N}\alpha + \frac{1}{2N}x\beta^2 \end{pmatrix}. \tag{5.44}$$

On the other hand, the first formula, $A = M(1)\sigma_3 M(1)^{-1}$, of (5.9) implies that $\det A = -1$. Thus (5.44) yield the identity

$$-(N - \alpha - \frac{1}{2}x\beta^2)^2 - \frac{1}{4}x^2\beta^2 + (\beta + \frac{1}{2}x(\alpha\beta - \gamma))^2 = -N^2. \tag{5.45}$$

By removing γ and β using (5.7), we obtain

$$-(N - \alpha - x\alpha')^2 - \frac{1}{2}x^2\alpha' + \frac{1}{2\alpha'}(2\alpha' + x\alpha'')^2 = -N^2. \tag{5.46}$$

This becomes (5.10) if we set $\theta = -\frac{1}{2}x\alpha$.

(vii) We take Γ as the union of circle of radius ϵ, centered at 1 and -1. (We have freedom to pick a contour.) Then from the formula of V, it is direct to check that

$$\|I - V^{-1}\|_{L^\infty(\Gamma)} \le \frac{(2+\epsilon)^N}{\epsilon^N} e^{-(1-\epsilon)x/2}. \tag{5.47}$$

Also by (5.19),

$$\|C_V\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \|C_+\|_{L^2(\Gamma)\to L^2(\Gamma)} \|I - V^{-1}\|_{L^\infty(\Sigma^M)} \le c_1 \|I - V^{-1}\|_{L^\infty(\Gamma)} \tag{5.48}$$

for some constant $c_1 > 0$. Here c_1 can be taken to be independent of $0 < \epsilon < 1/4$ from a simple scaling argument. Hence for large enough x, $\|(1 - C_V)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le 1/2$, and from (5.20),

$$M(w) = I + \frac{1}{2\pi i}\int_\Gamma \frac{(I - V^{-1})(s)}{s - w} ds + O\Big(\frac{\|I - V^{-1}\|^2_{L^\infty(\Gamma)} |\Gamma|}{dist(w, \Gamma)} \Big), \tag{5.49}$$

where $|\Gamma|$ is the size of Γ. Also, we have

$$\begin{aligned} M_1 &= -\frac{1}{2\pi i}\int_\Gamma (I + (1 - C_V)^{-1}(C_V I))(s)(I - V^{-1})(s) ds \\ &= -\frac{1}{2\pi i}\int_\Gamma (I - V^{-1})(s) ds - \frac{1}{2\pi i}\int_\Gamma (C_V I)(s)(I - V^{-1})(s) ds \\ &\quad - \frac{1}{2\pi i}\int_\Gamma ((1 - C_V)^{-1}(C_V(C_V I)))(s)(I - V^{-1})(s) ds, \end{aligned} \tag{5.50}$$

and the second and the third integrals on the last line are of order

$$O\big(\|I - V^{-1}\|^2_{L^\infty(\Gamma)}\big), \qquad O\big(\|I - V^{-1}\|^3_{L^\infty(\Gamma)}\big), \tag{5.51}$$

respectively. Therefore, we obtain

$$\begin{aligned} \beta = (M_1)_{12} &= -\frac{1}{2\pi i}\int_\Gamma (I - V^{-1})_{12}(s) ds + O\big(\|I - V^{-1}\|^2_{L^\infty(\Gamma)}\big) \\ &= \frac{1}{2\pi i}\int_{|s-1|=\epsilon} \Big(\frac{1+s}{1-s}\Big)^N e^{-\frac{1}{2}xs} ds + O(e^{-(1-\epsilon)x}) =: f_N(x) + O(e^{-(1-\epsilon)x}). \end{aligned} \tag{5.52}$$

After the change of variables $\frac{1}{2}x(s-1) = z$, we have

$$f_N(x) = \frac{2(-1)^N e^{-\frac{1}{2}x}}{2\pi i} \int_{|z|=\frac{1}{2}x\epsilon} \big(1+\frac{x}{z}\big)^{N-1} e^{-z}\big(1+\frac{z}{x}\big)\frac{dz}{z}. \tag{5.53}$$

This is precisely an integral representation of the Laguerre polynomial $L_{N-1}^{(1)}(x)$ (see e.g., [1] 22.10.8), and we find

$$\beta = 2(-1)^N e^{-\frac{1}{2}x} L_{N-1}^{(1)}(x) + O(e^{-(1-\epsilon)x}). \tag{5.54}$$

Similarly, from (5.50), we have

$$\begin{aligned} \alpha = (M_1)_{22} &= -\frac{1}{2\pi i}\int_\Gamma ((C_V I)(s)(I-V^{-1})(s))_{22} ds + O(\|I-V^{-1}\|^3_{L^\infty(\Gamma)}) \\ &= \frac{1}{(2\pi i)^2}\int_{\Gamma_1} ds \int_{\Gamma_2} dt \frac{\Phi(t)^{-1}\Phi(s)}{t-s} + O(e^{-3/2(1-\epsilon)x}) =: g_N(x) + O(e^{-3/2(1-\epsilon)x}). \end{aligned} \tag{5.55}$$

By a direct computation using (5.1), we find that

$$g_N'(x) = \frac{1}{2(2\pi i)^2}\int_{\Gamma_1} ds \int_{\Gamma_2} dt \bigg(\frac{1-t}{1+t}\bigg)^N e^{\frac{1}{2}xt}\bigg(\frac{1+s}{1-s}\bigg)e^{-\frac{1}{2}xs} = \frac{1}{2}(f_N(x))^2. \tag{5.56}$$

Since $\lim_{x\to\infty} g_N(x) = 0$, we find that

$$g_N(x) = \int_\infty^x \frac{1}{2}(f_N(y))^2 dy, \tag{5.57}$$

which proves (5.12).

For $w \in \mathbb{C}$ such that $dist(w,\Gamma) \geq \frac{1}{2}\epsilon$, the result (5.14) follows from (5.49) and (5.47), (5.48). For w such that $dist(w,\Gamma) < \frac{1}{2}\epsilon$, we algebraically transform the RHP so that the contour Γ are now the unions of circles of radius 2ϵ, centered at $1, -1$, and then apply the same estimates. By undoing the algebraic transformation and using the Cauchy's theorem, we obtain the result (5.14).

(viii) From the properties (ii), $M(0)$ is of the form

$$M(0) = \begin{pmatrix} A(x) & B(x) \\ B(x) & A(x) \end{pmatrix} \tag{5.58}$$

for some real function A and B such that $A^2 - B^2 = 1$. From the differential equation (5.6) when $w = 0$, we find

$$A' = \frac{1}{2}\beta B, \qquad B' = \frac{1}{2}\beta A. \tag{5.59}$$

Thus we have

$$A(x) + B(x) = (A(x_0)+B(x_0))e^{\int_{x_0}^x \frac{1}{2}\beta(s)ds}, \tag{5.60}$$

$$A(x) - B(x) = (A(x_0)-B(x_0))e^{-\int_{x_0}^x \frac{1}{2}\beta(s)ds}. \tag{5.61}$$

The asymptotics (5.11) and (5.14) then yield that

$$A(x) + B(x) = e^{\int_\infty^x \frac{1}{2}\beta(s)ds}, \tag{5.62}$$

$$A(x) - B(x) = e^{-\int_\infty^x \frac{1}{2}\beta(s)ds}, \tag{5.63}$$

which imply (5.16)

□

In Proposition 5.1 (i), the solution to the RHP exists for $x > 0$. Indeed it can be shown that the solution ceases to exist when $x = 0$ for $N \in \mathbb{N}$.

Lemma 5.2. *There is no solution to RHP* (5.3) *when* $x = 0$ *for* $N \in \mathbb{N}$.

Proof. When $x = 0$, $\Psi(w) = (\frac{1+w}{1-w})^N$. Suppose there is a solution M to the RHP (5.3). Let $L(w)$ be the matrix defined by

$$L(w) = \begin{cases} M(w)\begin{pmatrix} 1 & -(\frac{1+w}{1-w})^N \\ 0 & 1 \end{pmatrix}, & w \in \Omega_1, \\ M(w)\begin{pmatrix} 1 & 0 \\ -(\frac{1-w}{1+w})^N & 1 \end{pmatrix}, & w \in \Omega_2, \\ M(w), & w \in \Omega_0. \end{cases} \tag{5.64}$$

Then $L(w)$ is analytic in $\mathbb{C}$ except that its second column has a pole of order N at $w = 1$ and its first column has a pole of order N at $w = -1$. Since $L(w) \to I$ as $w \to \infty$, we find that L has the form

$$L(w) = I + \sum_{j=1}^{N} \frac{1}{(1-w)^j}\begin{pmatrix} 0 & a_j \\ 0 & b_j \end{pmatrix} + \sum_{k=1}^{N} \frac{1}{(1+w)^k}\begin{pmatrix} c_k & 0 \\ d_k & 0 \end{pmatrix}, \tag{5.65}$$

for some constants a_j, b_j, c_k, d_k. From (5.64) and (5.65), for $w \in \Omega_1$,

$$M(w) = L(w)\begin{pmatrix} 1 & (\frac{1+w}{1-w})^N \\ 0 & 1 \end{pmatrix}, \tag{5.66}$$

and especially the 12-entry is

$$M_{12}(w) = (\frac{1+w}{1-w})^N + \sum_{j=1}^{N} \frac{a_j}{(1-w)^j} + \sum_{k=1}^{N} \frac{c_k(1+w)^{N-k}}{(1-w)^N}. \tag{5.67}$$

From (5.3), M, hence M_{12}, is analytic at $w = 1$. By expanding (5.67) as a Laurent series around $w = 1$, and computing the coefficient of $(1-w)^{-1}$, we obtain the condition

$$-(-1)^N 2N + a_1 - (-1)^N c_1 = 0. \tag{5.68}$$

Similarly, for $w \in \Omega_2$,

$$M_{11}(w) = 1 + \sum_{j=1}^{N} \frac{a_j(1-w)^{N-j}}{(1+w)^N} + \sum_{k=1}^{N} \frac{c_k}{(1+w)^k}, \tag{5.69}$$

and by expanding $M_{11}(w)$ as a Laurant series around $w=-1$, and calculating the coefficient of $(1+w)^{-1}$, we obtain the second condition

$$-(-1)^N a_1 + c_1 = 0. \tag{5.70}$$

But there is no such a_1, c_1 satisfying (5.68) and (5.70) when $N \in \mathbb{N}$, and this is a contradiction. Therefore, there is no solution M to the RHP (5.3). □

6 Asymptotic analysis of orthogonal polynomials and the proof of Theorem 1.3

Set

$$t = \sqrt{q}. \tag{6.1}$$

As mentioned earlier, Theorem 1.3 is obtained by taking the limit (3.1) of the formula Lemma 4.2. For that purpose, we need asymptotics of π_k and N_k. In view of Lemma 3.1 and Lemma 4.2, we set

$$t = 1 - \frac{1}{2L}, \qquad k = [xL], \tag{6.2}$$

and take the limit $L \to \infty$ in this section. The results are summarized in Proposition 6.4 and Proposition 6.6 below. The asymptotics of $\Delta_n^{\pm\pm}$ are in Proposition 6.5, and the proof of Theorem 1.3 is given at the end of this section.

Let Σ be the unit circle $\{|z|=1\}$ in the complex plane, oriented counter-clockwise. Let Y be the solution to the following 2×2 Riemann-Hilbert problem (RHP) :

$$\begin{cases} Y(w) \text{ is analytic in } w \in \mathbb{C}\setminus\Sigma, \text{ and continuous up to the boundary,} \\ Y_+(w) = Y_-(w)\begin{pmatrix} 1 & w^{-k}(1+tw)^N(1+tw^{-1})^N \\ 0 & 1 \end{pmatrix}, \qquad w\in\Sigma, \\ Y(w)z^{-k\sigma_3} = I + O(w^{-1}), \qquad \text{as } w\to\infty. \end{cases} \tag{6.3}$$

Then due to the work of Fokas, Its and Kitaev ([12], see also [4]), the orthogonal polynomials π_k and its norm N_k of (4.8) are given by

$$\pi_k(w) = Y_{11}(w), \qquad N_k = Y_{12}(0), \qquad N_{k-1} = -Y_{21}(0)^{-1}. \tag{6.4}$$

The goal of this section is to find the asymptotics of Y with precise error bound as $L\to\infty$ with (6.2), and hence to find the asymptotics of π_k and N_k. We use the steepest-descent method for RHP, which was introduced by Deift and Zhou [11]. Throughout this section, N is a fixed parameter, while t and k would vary as L varies.

Define $m^{(1)}$ by

$$m^{(1)}(z) = (-1)^{\frac{k}{2}\sigma_3}Y(z)\begin{pmatrix}(1+tz)^N & 0 \\ 0 & (1+tz)^{-N}\end{pmatrix}(-1)^{\frac{k}{2}\sigma_3}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \qquad |z|<1, \tag{6.5}$$

and by

$$m^{(1)}(z) = (-1)^{\frac{k}{2}\sigma_3} Y(z) \begin{pmatrix} z^{-k}(1+tz^{-1})^N & 0 \\ 0 & z^k(1+tz^{-1})^{-N} \end{pmatrix} (-1)^{-\frac{k}{2}\sigma_3}, \qquad |z| > 1. \tag{6.6}$$

Then $m^{(1)}$ solves the new RHP

$$\begin{cases} m_+^{(1)}(z) = m_-^{(1)}(z) \begin{pmatrix} 1 & -\varphi(z) \\ \varphi(z)^{-1} & 0 \end{pmatrix}, & z \in \Sigma, \\ m^{(1)}(z) = I + O(z^{-1}), & \text{as } z \to \infty, \end{cases} \tag{6.7}$$

where

$$\varphi(z) := (-z)^k (1+tz)^N (1+tz^{-1})^{-N}. \tag{6.8}$$

This RHP is algebraically equivalent to the RHP (6.3).

Fix $0 < a < 1$. Let $\Sigma_1 = \{|z + 1 - \frac{1}{2L}| = \frac{a}{2L}\}$ and $\Sigma_2 = \{|z + 1 + \frac{1}{2L}| = \frac{a}{2L}\}$. We orient the circle Σ_1 counter-clockwise, and orient the circle Σ_2 clockwise. Note that we have plenty of freedom for the choice of the contour. Since $0 < a < 1$, Σ_1, Σ_2 and $|z| = 1$ have no intersection, and the complex plane is divided into four connected regions (see Figure 2). When $L > \frac{1+a}{2a}$, which we assume hereafter, the

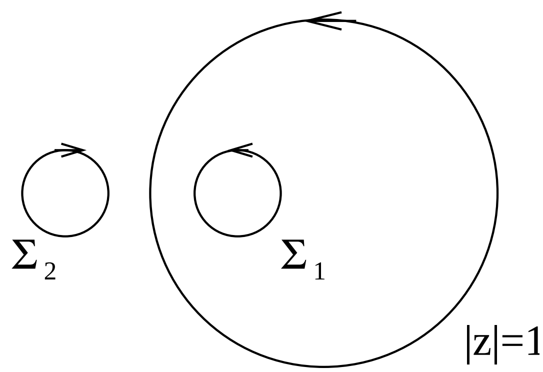

Figure 2: Contours Σ_1 nd Σ_2

point $-t^{-1}$ is inside the disk bounded by Σ_2. Define $m^{(2)}(z)$ by

$$\begin{cases} m^{(2)}(z) = m^{(1)}(z) \begin{pmatrix} 1 & \varphi(z) \\ 0 & 1 \end{pmatrix}, & \text{for } z \text{ between } \Sigma_1 \text{ and } |z| = 1, \\ m^{(2)}(z) = m^{(1)}(z) \begin{pmatrix} 1 & 0 \\ \varphi(z)^{-1} & 1 \end{pmatrix}, & \text{for } z \text{ in the unbounded component}, \\ m^{(2)}(z) = m^{(1)}(z), & \text{for } z \text{ inside } \Sigma_1 \text{ and } \Sigma_2. \end{cases} \tag{6.9}$$

Since $\varphi(z)$ is analytic in $\mathbb{C} \setminus \{-t\}$, and $\varphi(z)^{-1}$ is analytic in $\mathbb{C} \setminus \{-t^{-1}\}$, we find that $m^{(2)}(z)$ is analytic in all four regions. Moreover, by the jump condition of $m^{(1)}$ on $|z| = 1$, $m^{(2)}$ is analytic on $|z| = 1$.

Thus $m^{(2)}$ solves the following RHP :

$$\begin{cases} m^{(2)}(z) \text{ is analytic in } \mathbb{C}\setminus(\Sigma_1\cup\Sigma_2), \\ m_+^{(2)}(z) = m_-^{(2)}(z)v^{(2)}(z), \qquad z\in\Sigma_1\cup\Sigma_2, \\ m^{(2)}(z) = I + O(z^{-1}), \qquad \text{as } z\to\infty, \end{cases} \tag{6.10}$$

where

$$v^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & -\varphi(z) \\ 0 & 1 \end{pmatrix}, & z\in\Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ \varphi(z)^{-1} & 0 \end{pmatrix}, & z\in\Sigma_2. \end{cases} \tag{6.11}$$

Now we take the scaling

$$w := 2L(z+1). \tag{6.12}$$

Under this map $z\mapsto w$, Σ_1 and Σ_2 are mapped to $\Gamma_1 := \{w : |w-1| = a\}$ and $\Gamma_2 := \{w : |w+1| = a\}$, respectively. Set $\Gamma = \Gamma_1\cup\Gamma_2$. We define

$$m^{(3)}(w) := m^{(2)}(-1+\frac{w}{2L}). \tag{6.13}$$

Then $m^{(3)}$ solves the RHP

$$\begin{cases} m^{(3)}(w) \text{ is analytic in } w\in\mathbb{C}\setminus\Gamma, \\ m_+^{(3)}(w) = m_-^{(3)}(w)v^{(3)}(w), \qquad w\in\Gamma, \\ m^{(3)}(z) = I + O(w^{-1}), \qquad \text{as } w\to\infty, \end{cases} \tag{6.14}$$

where

$$v^{(3)}(w) = m^{(3)}(-1+\frac{w}{2L}). \tag{6.15}$$

Now we need the following Lemma.

Lemma 6.1. *For fixed $0 < a < \frac{2}{3}$, where a is the radius of Γ_1 and Γ_2, we have the following results.*

(1) With (6.2), for any $x_0 > 0$, there are positive constants C_1, c_1, L_1 such that

$$\|\varphi(-1+\frac{w}{2L}) - \Phi(w)\|_{L^\infty(\Gamma_1)} \le \frac{C_1}{L}e^{-c_1x}, \quad \|\varphi(-1+\frac{w}{2L}) - \Phi(w)\|_{L^\infty(\Gamma_2)} \le \frac{C_1}{L}e^{-c_1x} \tag{6.16}$$

for all $L\ge L_1$ and $x\in[x_0,\infty)$.

(2) For any $x_0 > 0$, there is a constant $C_2 > 0$ such that

$$\|M_+\|_{L^\infty(\Gamma)} \le C_2, \quad \|M_+^{-1}\|_{L^\infty(\Gamma)} \le C_2, \tag{6.17}$$

for all $x\in[x_0,\infty)$.

Proof. (1) By an elementary calculation, we have for any complex numbers f, g

$$|e^f - e^g| \le |f-g| e^{\max(Re(f), Re(g))}. \tag{6.18}$$

For $w \in \Gamma_1$, with (6.2),

$$\left|\varphi(-1+\frac{w}{2L}) - \Phi(w)\right| = \left|(1-\frac{w}{2L})^{[xL]+N} - e^{-\frac{1}{2}xw}\right| \left|\frac{1+w}{1-w}\right|^N. \tag{6.19}$$

Setting

$$f = \frac{[xL]+N}{xL} \log(1-\frac{w}{2L}), \qquad g = -\frac{w}{2L} \tag{6.20}$$

for (6.18), the difference (6.19) is less than or equal to

$$(\frac{2+a}{a})^N |e^{xLf} - e^{xLg}|. \tag{6.21}$$

Now with $k' := \frac{[xL]+N}{xL}$ and $\epsilon := \frac{w}{2L}$,

$$\begin{aligned} f - g = k' \log(1-\epsilon) + \epsilon &= \int_0^1 \frac{d}{ds}(k' \log(1-s\epsilon) + s\epsilon) ds \\ &= \int_0^1 \frac{-(k'-1)\epsilon - s\epsilon^2 + (k'-1)s\epsilon\bar{\epsilon} + s^2\epsilon^2\bar{\epsilon}}{|1-s\epsilon|^2} ds. \end{aligned} \tag{6.22}$$

An elementary calculation yields that $|arg(w)| \le \frac{\pi}{6}$ for $w \in \Gamma_1$ as $0 < a < \frac{2}{3}$, and hence we have $|arg(\epsilon)| \le \frac{\pi}{6}$ and $|arg(\epsilon^2)| \le \frac{\pi}{3}$. Also, when $L \ge \frac{\sqrt{2}}{\sqrt{2}-1} \ge \frac{a+1}{\sqrt{2}(\sqrt{2}-1)}$, we have $\frac{1}{2} \le |1-s\epsilon|^2 \le 2$ for $w \in \Gamma_1$. Hence noting that $k'-1 \ge 0$, we obtain

$$\begin{aligned} Re(f-g) &\le \int_0^1 \big\{-\frac{1}{2}(k'-1)Re(\epsilon) - \frac{1}{2} s Re(\epsilon^2) + 2(k'-1)s|\epsilon|^2 + 2s^2|\epsilon|^2 Re(\epsilon)\big\} ds \\ &= -\frac{1}{2}(k'-1)Re(\epsilon) - \frac{1}{4}Re(\epsilon^2) + (k'-1)|\epsilon|^2 + \frac{2}{3}|\epsilon|^2 Re(\epsilon). \end{aligned} \tag{6.23}$$

The first term satisfies $-\frac{1}{2}(k'-1)Re(\epsilon) \le 0$. For the third term, when $L \ge \frac{16N}{x_0}$, we have $k'-1 = \frac{[xL]-xL+N}{xL} \le \frac{N}{x_0 L} \le \frac{1}{16}$, and hence we find $(k'-1)|\epsilon|^2 \le \frac{1}{8}Re(\epsilon^2)$ as $|arg(\epsilon^2)| \le \frac{\pi}{3}$. It is also direct to check that the forth term has the estimate $\frac{2}{3}|\epsilon|^2 Re(\epsilon) \le \frac{1}{16}Re(\epsilon^2)$ if we take $L \ge \frac{64}{3}$. These estimates yield

$$Re(f-g) \le -\frac{1}{16}Re(\epsilon^2) < 0 \tag{6.24}$$

when $L \ge \max(\frac{64}{3}, \frac{16N}{x_0})$.

Similarly, from (6.22), using $|\epsilon| \le \frac{1+a}{L} \le \frac{2}{L}$ and $|1-s\epsilon|^2 \ge \frac{1}{2}$ when $L \ge \frac{\sqrt{2}}{\sqrt{2}-1}$, we obtain

$$\begin{aligned} |f-g| &\le 2\int_0^1 \big\{(k'-1)|\epsilon| + s|\epsilon|^2 + (k'-1)|\epsilon|^2 + s^2|\epsilon|^3\big\} ds \\ &\le (\frac{4N}{x_0} + 4)\frac{1}{L^2} + (\frac{4N}{x_0} + \frac{16}{3})\frac{1}{L^3} \\ &\le 3(\frac{4N}{x_0} + 4)\frac{1}{L^2}. \end{aligned} \tag{6.25}$$

Now the first inequality of (6.16) on Γ_1 is obtained using (6.21). The second inequality of (6.16) on Γ_2 follows from the inequality on Γ_1 and the symmetry under $w \mapsto -\overline{w}$.

(2) From (5.20) and (5.19), we have $M_+ = I + (1 - C_V)^{-1}(C_V I)$. By Proposition 5.1, $(1 - C_V)^{-1}$ exists for all $x > 0$, and also it is easy to check that C_V and $(1 - C_V)^{-1}$ are continuous in x. Hence from (5.20), M_+ is uniformly bounded for x in a compact subset of $(0, \infty)$. Also, from the analysis of Proposition 5.1 (vii), C_V and $(1 - C_V)^{-1}$ are indeed bounded as $x \to \infty$. Hence we obtain the uniform boundedness of M_+ for $x \in [x_0, \infty)$. The boundedness of M_+^{-1} follows from the boundedness of M_+, together with the fact that $\det M_+ = 1$. ☐

Set

$$R(w) := m^{(3)}(w)(M(w))^{-1}. \tag{6.26}$$

Then R solves the RHP,

$$\begin{cases} R_+(w) = R_-(w)v_R(w), & w \in \mathbb{C} \setminus \Sigma, \\ R(w) = I + O(w^{-1}), & \text{as } w \to \infty, \end{cases} \tag{6.27}$$

where

$$v_R = M_- v^{(3)} V^{-1} (M_-)^{-1} = M_+ V^{-1} v^{(3)} M_+^{-1}. \tag{6.28}$$

Then v_R shares the following properties.

Lemma 6.2. *Let C_{v_R} be the operator introduced in* (5.19) *for v_R. For any fixed $x_0 > 0$, there are positive constants C_3, c_3, L_3 such that*

$$\|I - v_R^{-1}\|_{L^\infty(\Gamma)} \leq \frac{C_3}{L} e^{-c_3 x}, \tag{6.29}$$

uniformly in $x \in [x_0, \infty)$ for $L \geq L_3$. Also for L and x in the same range, the operator C_{v_R} acting on $L^2(\Gamma)$ satisfies

$$\|(1 - C_{v_R})^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq 2, \tag{6.30}$$

and the matrix R in (6.26) *is given by*

$$\begin{aligned} R(w) &= m^{(3)}(w)(M(w))^{-1} \\ &= I + \frac{1}{2\pi i}\int_\Gamma \frac{I - v_R(s)^{-1}}{s - w} ds + \frac{1}{2\pi i}\int_\Gamma \frac{((1 - C_{v_R})^{-1} C_{v_R} I)(s)(I - v_R(s)^{-1})}{s - w} ds \end{aligned} \tag{6.31}$$

Proof. Since

$$I - (v^{(3)})^{-1} V = \begin{pmatrix} 0 & \Phi(w) - \varphi(-1 + \frac{w}{2L}) \\ 0 & 0 \end{pmatrix}, \tag{6.32}$$

for $w \in \Gamma_1$, from Lemma 6.1 (1), we obtain for $L \geq L_1$,

$$\|I - (v^{(3)})^{-1} V\|_{L^\infty(\Gamma_1)} \leq \frac{C_1}{L} e^{-c_1 x}. \tag{6.33}$$

A similar calculation yields the same bound for Γ_2. Therefore using Lemma 6.1, we obtain for $L \geq L_1$,

$$\|I - v_R^{-1}\|_{L^\infty(\Gamma)} = \|M_+(I - (v^{(3)})^{-1} V) M_+^{-1}\|_{L^\infty(\Gamma)} \leq 2C_1 C_2^2 \frac{1}{L} e^{-c_1 x}, \tag{6.34}$$

and we obtain (6.29) (with $c_3 = c_1$). For (6.30), note that since the Cauchy operator C_+ on $L^2(\Gamma)$ is bounded, we have

$$\|C_{v_R}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \frac{C'}{L} e^{-c_3 x} \tag{6.35}$$

for $L \geq L_1$ with some new constant $C' > 0$. But if we take L large enough so that $\frac{C'}{L} e^{-c_3 x} \leq \frac{C'}{L} \leq \frac{1}{2}$, we have

$$\|C_{v_R}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \frac{1}{2} \tag{6.36}$$

uniformly in $x \in [x_0, \infty)$. Now the result for C_{v_R} follows from the Neumann series, and (6.27) follows from the general theory of RHP (cf. proof of Proposition 5.1 (i)). □

Under the sequence of transformations $Y \to m^{(1)} \to m^{(2)} \to m^{(3)}$, the quantities (6.4) of interest become

$$N_k = m_{11}^{(3)}(2L), \qquad N_{k-1}^{-1} = m_{22}^{(3)}(2L), \qquad \pi_l(0) = -(-1)^k m_{12}^{(3)}(2L). \tag{6.37}$$

From (6.31), using (6.29) and (6.30), we find that for x, L as in Lemma 6.2,

$$|m^{(3)}(2L) M(2L)^{-1} - I| \leq \frac{C'}{L^2} e^{-c_3 x}, \tag{6.38}$$

for some constant $C' > 0$.

Now the following estimates for $M(2L)$ follows from (5.20) in the proof of Proposition 5.1 (i) and also the proof of Lemma 6.1 (ii).

Lemma 6.3. *For any fixed $x_0 > 0$, let c_3 and L_3 be as in Lemma 6.2. There are positive numbers C_4 and C_5 such that*

$$\left| M(2L) - I - \frac{M_1}{2L} \right| \leq \frac{C_4}{L^2} e^{-c_3 x}, \qquad |M(2L)| \leq C_5, \tag{6.39}$$

for all $x \in [x_0, \infty)$ and $L \geq L_3$.

Thus (6.38) yields that

$$\left| m^{(3)}(2L) - I - \frac{M_1}{2L} \right| \leq \frac{C_6}{L^2} e^{-c_3 x}, \tag{6.40}$$

for some constant $C_6 > 0$. From this and (6.37), we obtain the asymptotics of N_k and $\pi_k(0)$:

Proposition 6.4. *With (6.2), for any fixed $x_0 > 0$, there are positive constants C_6, c_6, L_6 such that*

$$\big|N_k - 1 + \frac{\alpha(x,N)}{2L}\big| \le \frac{C_6}{L^2}e^{-c_6 x}, \tag{6.41}$$

$$\big|N_{k-1}^{-1} - 1 - \frac{\alpha(x,N)}{2L}\big| \le \frac{C_6}{L^2}e^{-c_6 x}, \tag{6.42}$$

$$\big|\pi_k(0) + (-1)^k\frac{\beta(x,N)}{2L}\big| \le \frac{C_6}{L^2}e^{-c_6 x}, \tag{6.43}$$

where α and β are defined in Proposition 5.1 (iii).

As in the proof Lemma 7.1 of [4], it is direct to obtain the following result from Proposition 6.4, and we skip the proof.

Proposition 6.5. *For any fixed $x_0 > 0$, there are positive constants L_7, c_7 such that when $2n = [xL]$,*

$$\begin{aligned}
\prod_{j\ge n} N_{2j+2}^{-1} &= \exp\Big(\int_x^\infty \frac{1}{4}\alpha(y,N)dy\Big)\big(1 + O(\frac{1}{L}e^{-c_7 x})\big), & (6.44)\\
\prod_{j\ge n} N_{2j+1}^{-1} &= \exp\Big(\int_x^\infty \frac{1}{4}\alpha(y,N)dy\Big)\big(1 + O(\frac{1}{L}e^{-c_7 x})\big), & (6.45)\\
\prod_{j\ge n}(1 \pm \pi_{2j+2}(0)) &= \exp\Big(\mp\int_x^\infty \frac{1}{4}\beta(y,N)dy\Big)\big(1 + O(\frac{1}{L}e^{-c_7 x})\big), & (6.46)\\
\prod_{j\ge n}(1 \mp \pi_{2j+1}(0)) &= \exp\Big(\mp\int_x^\infty \frac{1}{4}\beta(y,N)dy\Big)\big(1 + O(\frac{1}{L}e^{-c_7 x})\big), & (6.47)
\end{aligned}$$

for $x \in [x_0, \infty)$ and $L \ge L_7$.

Also under the transformations $Y \to m^{(1)} \to m^{(2)} \to m^{(3)}$, ϕ_k in (4.10) is, with the relation $z = -1 + \frac{w}{2L}$, given by

$$\phi_k(z) = \begin{cases} -(-1)^k m_{12}^{(3)}(w), & w \in \Omega_1,\\ z^k(1+tz)^N(1+tz^{-1})^{-N}m_{11}^{(3)}(w), & w \in \Omega_2,\\ -(-1)^k m_{12}^{(3)}(w) + z^k(1+tz)^N(1+tz^{-1})^{-N}m_{11}^{(3)}(w), & w \in \Omega_0 \end{cases} \tag{6.48}$$

(recall Figure 1). Hence, under (6.2), from (6.31), for any fixed $x > 0$ and fixed $w \in \mathbb{C}$,

$$\lim_{L\to\infty} m^{(3)}(w) = M(w). \tag{6.49}$$

(For w near the contour Γ, we could take different radius a of the contours Γ_1, Γ_2.) Therefore, (6.48) has the limit

$$\lim_{L\to\infty}(-1)^k\phi_k(-1 + \frac{w}{2L}) = \begin{cases} -M_{12}(w) & w \in \Omega_1,\\ -M_{12}(w) + \Phi(w)M_{11}(w), & w \in \Omega_0,\\ \Phi(w)M_{11}(w), & w \in \Omega_2, \end{cases} \tag{6.50}$$

for each fixed $x > 0$ and $w \in \mathbb{C}$, where $k = [xL]$. On the other hand, by noting that $\phi_k^*(z) = z^k(1+\sqrt{q}z)^N(1+\sqrt{q}z^{-1})^{-N}\phi_k(z^{-1})$, we obtain from (6.50) that

$$\lim_{L\to\infty} \phi_k^*(-1+\frac{w}{2L}) = \begin{cases} M_{11}(-w) & w \in \Omega_1, \\ M_{11}(-w) - \Phi(w)M_{12}(-w), & w \in \Omega_0, \\ -\Phi(w)M_{12}(-w), & w \in \Omega_2. \end{cases} \tag{6.51}$$

Using the symmetry $M(w) = \sigma_1 M(-w)\sigma_1$ of Proposition 5.1 (ii), this is equal to

$$\lim_{L\to\infty} \phi_k^*(-1+\frac{w}{2L}) = \begin{cases} M_{22}(w) & w \in \Omega_1, \\ M_{22}(w) - \Phi(w)M_{21}(w), & w \in \Omega_0, \\ -\Phi(w)M_{21}(w), & w \in \Omega_2. \end{cases} \tag{6.52}$$

Define $M^{(1)}(w) = M^{(1)}(w; x, N)$ by

$$M^{(1)}(w) := \begin{cases} M(w) & w \in \Omega_1, \\ M(w)\begin{pmatrix} 1 & -\Phi(w) \\ 0 & 1 \end{pmatrix}, & w \in \Omega_0, \\ M(w)\begin{pmatrix} 1 & -\Phi(w) \\ \Phi(w)^{-1} & 0 \end{pmatrix}, & w \in \Omega_2 \setminus \{-1\}. \end{cases} \tag{6.53}$$

In other words, from the jump condition of the RHP (5.3), $M^{(1)}$ is obtained by taking analytic continuation of $M(w)$ for $w \in \Omega_1$. Then the results (6.50) and (6.52) can be written in a compact form :

Proposition 6.6. *We have with $k = [xL]$,*

$$\lim_{L\to\infty} (-1)^k \phi_k(-1+\frac{w}{2L}) = -M_{12}^{(1)}(w) \tag{6.54}$$

$$\lim_{L\to\infty} \phi_k^*(-1+\frac{w}{2L}) = M_{22}^{(1)}(w) \tag{6.55}$$

for each $x > 0$ and $w \in \mathbb{C}$.

Proof of Theorem 1.3

Now we prove Theorem 1.3. By combining Lemma 3.1, Lemma 4.2, Proposition 6.5 and Proposition 6.6, we obtain for $\rho > 0$ and $x > 0$,

$$\begin{aligned} &\mathbb{P}(H^{\boxslash}(N;\rho) \le x) \\ &= \frac{1}{2}\Big\{ \big[M_{22}^{(1)}(w) - M_{12}^{(1)}(w)\big](\mathrm{E_N}(x))^{-1} + \big[M_{22}^{(1)}(w) + M_{12}^{(1)}(w)\big]\,\mathrm{E_N}(x) \Big\}\,\mathrm{F_N}(x), \end{aligned} \tag{6.56}$$

where

$$w = \frac{2}{\rho} - 1. \tag{6.57}$$

Thus we set

$$\mathrm{a_N}(x,\rho)=M^{(1)}_{22}(\frac{2}{\rho}-1;x),\qquad \mathrm{b_N}(x,\rho)=M^{(1)}_{12}(\frac{2}{\rho}-1;x). \tag{6.58}$$

From the RHP for M, $\mathrm{a_N}(x,\rho),\mathrm{b_N}(x,\rho)$ are analytic in $x>0,\rho>0$, and $\mathrm{E_N}(x),\mathrm{F_N}(x)$ are analytic in $x>0$.

The properties (i), (ii) of Theorem 1.3 are given in Proposition 5.1 (iv), (vi) and (vii).

For the property (iii), we note that it is direct to check that when M satisfies the differential equations (vi), (v) of Proposition 5.1, $M^{(1)}$ also satisfies the *same* differential equations. This implies the property (iii).

The asymptotics (iv) of $\mathrm{a_N},\mathrm{b_N}$ follows from (5.14), and the definition of $M^{(1)}$.

The asymptotics (1.31) in (v) follows from the normalization condition of the RHP : $M(w)\to I$ as $w\to\infty$, and (1.32) is obtained from Proposition 5.1 (viii) and the definition of $M^{(1)}$.

Now we compute (1.33). With $x=y\rho$ and $w=\frac{2}{\rho}-1$, we have

$$\Phi(w;x)=e^{-y}e^{\frac12 y\rho}\frac{1}{(\rho-1)^N}=O(e^{\frac12 y\rho}). \tag{6.59}$$

As $\rho\to\infty$, using (1.28), with the change of variables $s=1-(2/\rho)u$,

$$\begin{aligned}\mathrm{a_N}(y\rho,\rho)&=\Phi(\frac{2}{\rho}-1;y\rho)\big(\Gamma_1(-\frac{2}{\rho}+1,y\rho)+O(e^{-(1-2\epsilon)y\rho})\big)\\&=\frac{1}{2\pi i}\int_{|u|=1/2}e^{y(u-1)}\big(\frac{\rho-u}{\rho-1}\big)^N\frac{du}{u^N(u-1)}+O(e^{-(\frac12-2\epsilon)y\rho})\\&=\frac{1}{2\pi i}\int_{|u|=1/2}e^{y(u-1)}\frac{du}{u^N(u-1)}\big(1+O(\rho^{-1})\big)+O(e^{-(\frac12-2\epsilon)y\rho}).\end{aligned} \tag{6.60}$$

But the function

$$g(y):=\frac{1}{2\pi i}\int_{|u|=1/2}e^{y(u-1)}\frac{du}{u^N(u-1)} \tag{6.61}$$

satisfies $g'(y)=e^{-y}\frac{y^{N-1}}{(N-1)!}$ and $g(0)=0$. Hence $g(y)=P(N,y)$, the incomplete Gamma function, and we obtain the first of (1.33). The second of (1.33) follows from (1.29) and (6.59).

7 Limiting distributions as $N\to\infty$

In Theorem 4.2 of [7], the authors computed the limiting distributions of the last passage percolation time $G^{\boxslash}(N;\alpha)$ of (3.4) as $N\to\infty$ for various values of α. Similar results are also obtained in [6] for a Poisson percolation model with a symmetry condition, which also has interpretations as the longest increasing subsequence of random involutions. In this section, we take similar limit for $H^{\boxslash}(N;\rho)$. The results are such that we take the formal limit Lemma 3.1 in Theorem 4.2 of [7] assuming that the two limits $N\to\infty$ and $L\to\infty$ interchange. The functions F_1,F_4 and $F^{\boxslash}$ are defined in [7, 6].

Theorem 7.1. *For each fixed $x \in \mathbb{R}$ and ρ,*

$$\lim_{N\to\infty} \mathbb{P}\Big(\frac{H^{\boxslash}(N;\rho) - 4N}{2^{4/3}N^{1/3}} \le x \Big) = \begin{cases} F_4(x), & 0 \le \rho < 2, \\ F_1(x), & \rho = 2, \\ 0, & \rho > 2. \end{cases} \tag{7.1}$$

Also, for each fixed $x \in \mathbb{R}$ and $w \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(\frac{H^{\boxslash}(N;\rho) - 4N}{2^{4/3}N^{1/3}} \le x \Big) = F^{\boxslash}(x;w), \qquad \rho = 2 - \frac{2^{5/3}w}{N^{1/3}} \tag{7.2}$$

From the formula in Theorem 1.3, this result can be obtained by applying the Deift-Zhou steepest-descent method to the Riemann-Hilbert problem (5.3). The analysis is analogous to that of [4, 6], and we do not provide any details here. Steepest-descent analysis for an extension of the RHP (5.3), which includes the analysis of the above Theorem 7.1 as a special case, will be carried out in a later publication, [3].

8 Correlation functions

In this last section, we give some remarks on the correlation functions for the general interpolating ensemble with the density (1.6) with $\beta = 1$,

$$p(\xi_1, \cdots, \xi_N; A; w) = \frac{1}{Z_A} \prod_{1\le i<j\le N} (\xi_i - \xi_j) \prod_{j=1}^{N} w(\xi_j) e^{A(-1)^j \xi_j}, \tag{8.1}$$

on $\mathbb{R}^N_{ord} := \{\xi_N \le \xi_{N-1} \le \cdots \le \xi_1\}$ (N is even). It has been known that the correlation functions for the orthogonal ensemble (when $A = 0$) and the symplectic ensemble (when $A = +\infty$) can be expressed in terms of the Pfaffian, or the square root of the determinant, of an antisymmetric matrix (see e.g., [22, 29]). For (8.1) with general A, Rains ([23]) computed the correlation functions and showed that it is again expressible in terms of Pfaffians but of a different matrix. In this section we remark that the result of Rains can be obtained from the argument of Tracy and Widom [29] after a minor change.

In [29], Tracy and Widom developed a systematic method to express various correlation functions of orthogonal and symplectic ensembles in terms of certain Fredholm determinants. Especially in Section 9 of [29], Tracy and Widom expressed the correlation functions of general orthogonal ensemble. The computation below for the above ensemble (8.1) is identical to that of Section 9 of [29] except the change of the asymmetric factor $\epsilon \to \epsilon_A$.

We assume that N is even in the below. When N is odd, one needs some change of the formulas. The starting point is the following identity in Remark 7.6.1, [5] :

$$e^{A\sum_{j=1}^{N}(-1)^j \xi_j} = \mathrm{Pf}\big(\mathrm{sgn}(\xi_j - \xi_k) e^{A|\xi_j - \xi_k|}\big)_{1\le j,k\le N}. \tag{8.2}$$

This can be checked by noting that the Pfaffian on the right-hand-side is the square root of the determinant D_N of the matrix $\big(\mathrm{sgn}(\xi_j - \xi_k) e^{A|\xi_j-\xi_k|}\big)_{1\le j,k\le N}$, and by finding the relation $D_N =$

$e^{2A(\xi_{N-1}-\xi_N)}D_{N-2}$ using proper row and column operations. From (8.2), for any bounded function f, we have

$$\begin{aligned} Z_A \cdot \int_{\mathbb{R}^N_{ord}} p(\xi_1, \cdots, \xi_N; A; w) \prod_{j=1}^N (1 + f(\xi_j)) d\xi_1 \cdots d\xi_N \\ = \int_{\mathbb{R}^N_{ord}} \mathrm{Pf}\big(\mathrm{sgn}(\xi_j - \xi_k) e^{A|\xi_j - \xi_k|}\big)_{1\le j,k\le N} \det\big(\xi_k^j w(\xi_k)(1 + f(\xi_k))\big)_{1\le j,k\le N} d\xi_1 \cdots d\xi_N \end{aligned} \tag{8.3}$$

We also need the following result of de Bruijn (equations (4.6)-(4.8) of [8]) : given a measure space (X, μ), for an anti-symmetric function $s(x, y) = -s(y, x)$ on $X \times X$,

$$\begin{aligned} \frac{1}{N!} \int_{X^N} \mathrm{Pf}(s(x_i, x_j))_{1\le i,j\le N} \det(\phi_i(x_j))_{1\le i,j\le N} d\mu(x_1) \cdots d\mu(x_N) \\ = \mathrm{Pf}\bigg(\int_X \int_X \phi_i(x) s(x, y) \phi_j(y) d\mu(x) d_\mu(y)\bigg)_{1\le i,j\le N} \end{aligned} \tag{8.4}$$

for a sequence of functions ϕ_j on X. This is a generalization of (1.4) of [29] where the authors took the special case when $s(x, y) = \mathrm{sgn}(x - y)$. Now with $s(x, y) = \mathrm{sgn}(x - y)e^{A|x-y|}$ and $\phi_j(x) = x^j w(x)(1 + f(x))$, the square of (8.3) is equal to

$$\det\bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{sgn}(x - y) e^{A|x-y|} x^j y^k w(x) w(y)(1 + f(x))(1 + f(y)) dx dy\bigg)_{0\le j,k\le N-1}. \tag{8.5}$$

Here the $N!$ term in (8.4) disappears since $\mathbb{R}^N_{ord}$ is the ordered set $\{\xi_N \le \cdots \le \xi_1\}$ and we take $X = \mathbb{R}$. This formula is same as the second displayed formula of section 9 of [29] with the modification that $\epsilon(x - y)$ (which is $\mathrm{sgn}(x - y)$) is replaced with $\mathrm{sgn}(x - y)e^{A|x-y|}$.

The rest of argument is same as section 9, [29] except that the operator ϵ whose kernel is $\mathrm{sgn}(x - y)$ in [29] is now changed to the operator ϵ_A, defined by

$$(\epsilon_A h)(x) = \int_{\mathbb{R}} \epsilon_A(x - y) h(y) dy, \qquad \epsilon_A(x - y) := \mathrm{sgn}(x - y) e^{A|x-y|}, \tag{8.6}$$

for a suitable class of functions h. With this modification, the result analogous to (3.3), (9.1) of [29] is the following.

Lemma 8.1. *(Theorem 1.1 of [23]) Set $\psi_j(x) = p_j(x)w(x)$ where $p_j(x)$, $j = 0, 1, \cdots$, is an arbitrary sequence of polynomials of exact degree j. Let M be the matrix*

$$M = \bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} \epsilon_A(x - y) \psi_j(x) \psi_k(y) dx dy\bigg)_{0\le j,k\le N-1} \tag{8.7}$$

and set $M^{-1} = (\mu_{jk})$. Then we have

$$\int_{\mathbb{R}^N_{ord}} p(\xi_1, \cdots, \xi_N; A) \prod_{j=1}^N (1 + f(\xi_j)) d\xi_1 \cdots d\xi_N = \sqrt{\det(1 + K_N^{(A)} f)} \tag{8.8}$$

where the operator $K_N^{(A)}$ has the 2×2 matrix kernel

$$K_N(x, y; A) = \begin{pmatrix} S_N(x, y; A) & S_N D(x, y; A) \\ IS_N(x, y; A) - \epsilon_A(x - y) & S_N(x, y; A) \end{pmatrix} \tag{8.9}$$

and

$$
\begin{aligned}
S_N(x,y;A) &= -\sum_{j,k=0}^{N-1} \psi_j(x)\mu_{jk}(\epsilon_A\psi_k)(y), & (8.10)\\
IS_N(x,y;A) &= -\sum_{j,k=0}^{N-1} (\epsilon_A\psi_j)(x)\mu_{jk}(\epsilon_A\psi_k)(y), & (8.11)\\
S_ND(x,y;A) &= \sum_{j,k=0}^{N-1} \psi_j(x)\mu_{jk}\psi_k(y). & (8.12)
\end{aligned}
$$

The matrix elements in the above determinant for general A is significantly simplified for the Laguerre case, $w(x) = e^{-x}1_{x\geq 0}$ by [15], which is an extension for general A of the results for Laguerre orthogonal and symplectic ensembles (see e.g. [2, 31]).

When $A = 0$, $\epsilon_A = \epsilon$ in the notation of [29], and the kernel (8.9) is equal to (9.1) of [29], which is the $\beta = 1$ orthogonal ensemble. On the other hand, when $A \to +\infty$, we do not have a proof that $K_N(x,y;A)$ converge to the kernel (8.1) of [29] for the $\beta = 4$ symplectic ensemble. However, we note that for smooth h which decays fast at $\pm\infty$, integrations by parts yield that

$$
\begin{aligned}
(\epsilon_A h)(x) &= -\frac{2}{A^2}h'(x) + \frac{1}{A^2}(\epsilon_A h'')(x)\\
&= -\frac{2}{A^2}h'(x) - \frac{2}{A^4}h^{(3)}(x) + \frac{1}{A^4}(\epsilon_A h^{(4)})(x) = \cdots .
\end{aligned} \tag{8.13}
$$

Thus when $A \to +\infty$, it seems that the main contribution to $(\epsilon_A h)$ comes from $-h'$. If we replace $(\epsilon_A\psi_k)$ in (8.10)-(8.12) above by $-\psi_k'$ and drop the term $\epsilon_A(x-y)$, (8.9) is equal to (8.1) of [29] if the notations IS_N and S_ND there are exchanged. This seems to be an indication that the kernel $K_N(x,y;A)$ actually converges to the kernel (8.1) of [29], the $\beta = 4$ symplectic ensemble, as $A \to +\infty$. Nevertheless, Proposition 1.1 shows that the determinant $\det(1+K_N^{(A)}f)$ converges to the corresponding determinant for $\beta = 4$ symplectic ensemble as $A \to \infty$ for a proper class of functions f.

We finish this section with some properties of the operator ϵ_A which can be checked easily. Let h be a function in the Schawrz class.

- Let $g = \epsilon_A h$. Then $g'' - A^2 g = 2h'$.
- If h, in addition to the smooth and decay conditions, satisfies $\int_{\mathbb{R}} h(x)ds = 0$, we have

$$
(\epsilon_A^{-1}h)(x) = \frac{1}{2}h'(x) - \frac{A^2}{2}\int_{-\infty}^{x} h(t)dt. \tag{8.14}
$$

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